4 | APPLICATIONS OF DERIVATIVES

Figure 4.1 As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Chapter Outline

Introduction

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward? (See **Example 4.3**.)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

4.1 | Related Rates

Learning Objectives

4.1.1 Express changing quantities in terms of derivatives.

4.1.2 Find relationships among the derivatives in a given problem.

4.1.3 Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 4.1

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\text{ cm}^3/\text{sec}$ (**Figure 4.2**). How fast is the radius increasing when the radius is 3 cm?

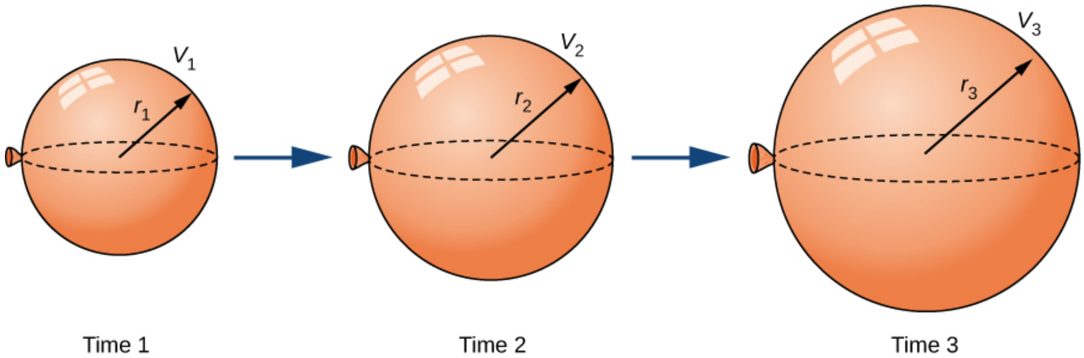


Figure 4.2 As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi[r(t)]^3 \text{ cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi[r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V'(t) = 2 \text{ cm}^3/\text{sec}$. Therefore,

$$2\text{cm}^3/\text{sec} = \left(4\pi[r(t)]^2 \text{ cm}^2\right) \cdot (r'(t)\text{cm/s}),$$

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2}\text{cm/sec}.$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi}\text{cm/sec}.$$

4.1 What is the instantaneous rate of change of the radius when $r = 6$ cm?

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

Example 4.2

An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

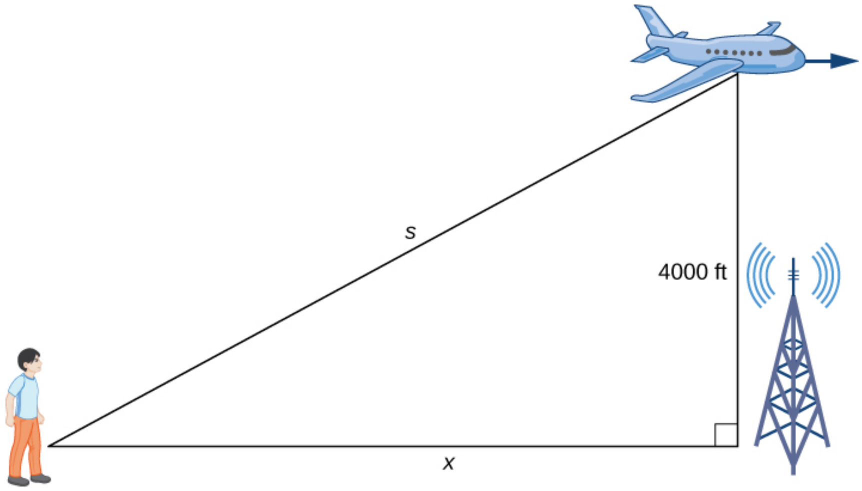


Figure 4.3 An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the Pythagorean theorem to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec.}$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0.$

4.2 What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

We now return to the problem involving the rocket launch from the beginning of the chapter.

Example 4.3

Chapter Opener: A Rocket Launch

Figure 4.4 (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

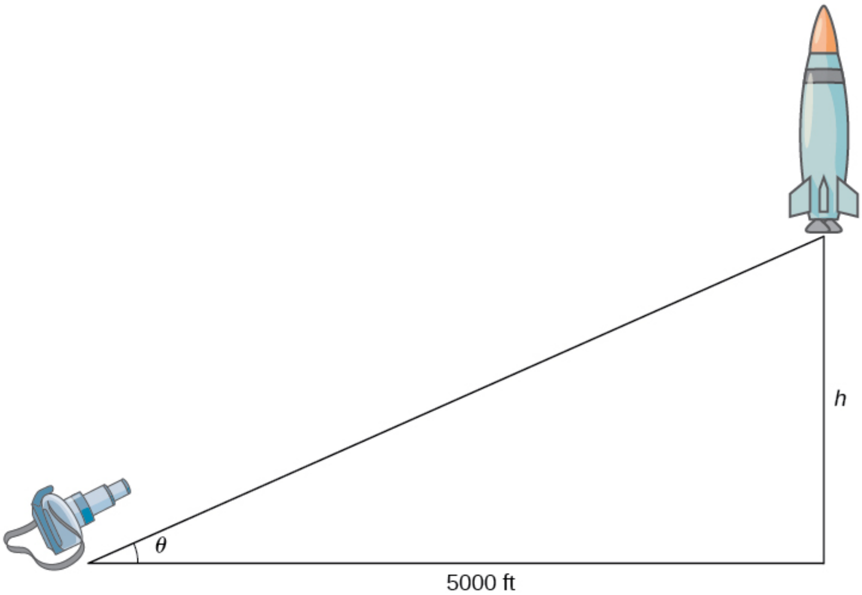


Figure 4.5 A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ, respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan\theta = \frac{h}{5000}.$$

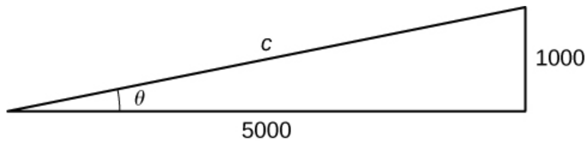
This gives us the equation

$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\text{ ft.}$$

Therefore, when $h = 1000$, we have

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

4.3 What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 4.4

Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\text{ ft}^3\text{/sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\text{ ft}$?

Solution

Step 1: Draw a picture introducing the variables.

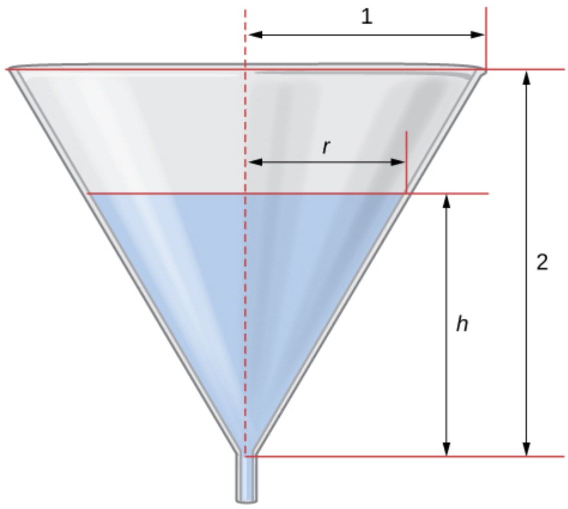


Figure 4.6 Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\ \text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\ \text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\ \text{ft/sec}.$$

4.4 At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

4.1 EXERCISES

For the following exercises, find the quantities for the given equation.

1. Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4.$

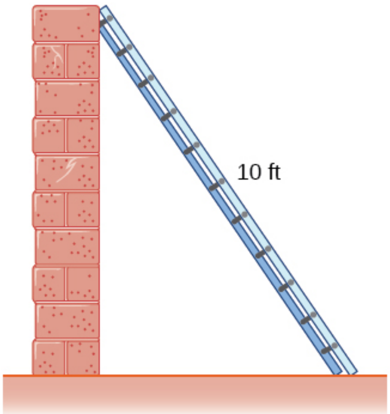
2. Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1.$

3. Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3.$

For the following exercises, sketch the situation if necessary and used related rates to solve for the quantities.

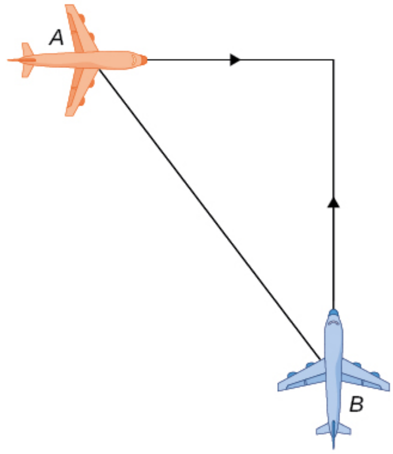
4. **[T]** If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$ If R_1 is increasing at a rate of $0.5\ \Omega/\text{min}$ and R_2 decreases at a rate of $1.1\Omega/\text{min},$ at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega$?

5. A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



6. A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

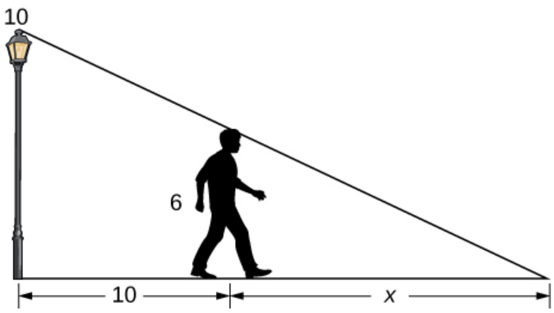
7. Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at $300\text{ mi/h}.$ If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



8. You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled $4\text{ mi},$ at what rate is the distance between you changing?

9. Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant $55\text{ mph},$ find the rate at which the distance between the buses is changing when they are 13 mi apart, heading toward each other.

10. A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of $3\text{ ft/sec}.$ What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11. Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

12. A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

13. Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

14. A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15. Using the previous problem, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

For the following exercises, draw and label diagrams to help solve the related-rates problems.

16. The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17. The volume of a cube decreases at a rate of 10 m^3/s. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

18. The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19. The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

20. The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21. The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

22. The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23. A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

24. A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of 4 cm^2/sec. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is 20 cm^2.

For the following exercises, consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

25. How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min?

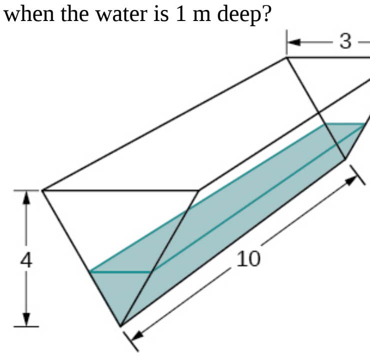
26. Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min.

27. If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

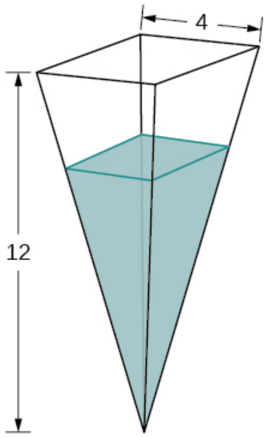
28. A vertical cylinder is leaking water at a rate of 1 ft^3/sec. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

29. A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

30. A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of 5 m^3/min. At what rate does the height of the water change when the water is 1 m deep?



31. A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}$ m/sec?



For the following problems, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of 25 ft^3/min. The radius of the pool is 10 ft.

32. Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

33. Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

34. If the height is increasing at a rate of 1 in./sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

35. Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of 10 ft^3/min. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

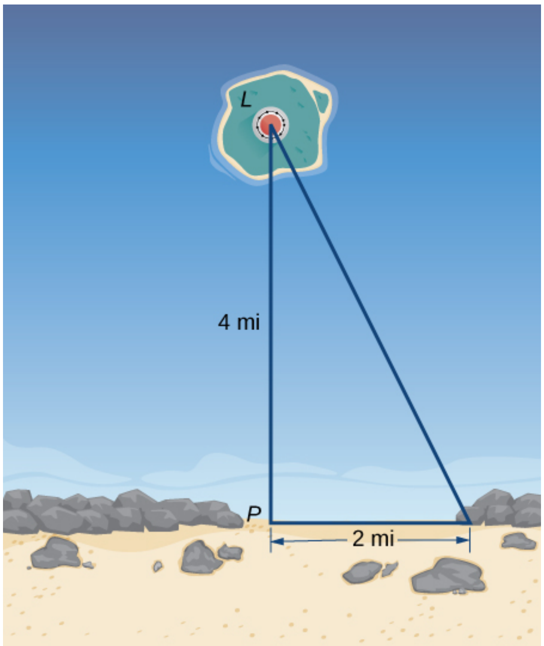
36. Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in./min.

For the following exercises, draw the situations and solve the related-rate problems.

37. You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

38. You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

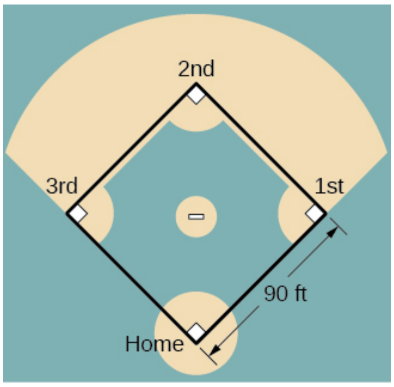
39. A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



40. Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

41. You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

For the following exercises, refer to the figure of baseball diamond, which has sides of 90 ft.



42. **[T]** A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the distance between the ball and the batter change when 2 sec have passed?

43. **[T]** A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (*Hint*: Recall the law of cosines.)

44. **[T]** A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

45. **[T]** Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

4.2 | Linear Approximations and Differentials

Learning Objectives

4.2.1 Describe the linear approximation to a function at a point.
4.2.2 Write the linearization of a given function.
4.2.3 Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
4.2.4 Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials **Introduction to Power Series and Functions (http://cnx.org/content/m53760/latest/)** .

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure 4.7(a) shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (**Figure 4.7**(b)). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

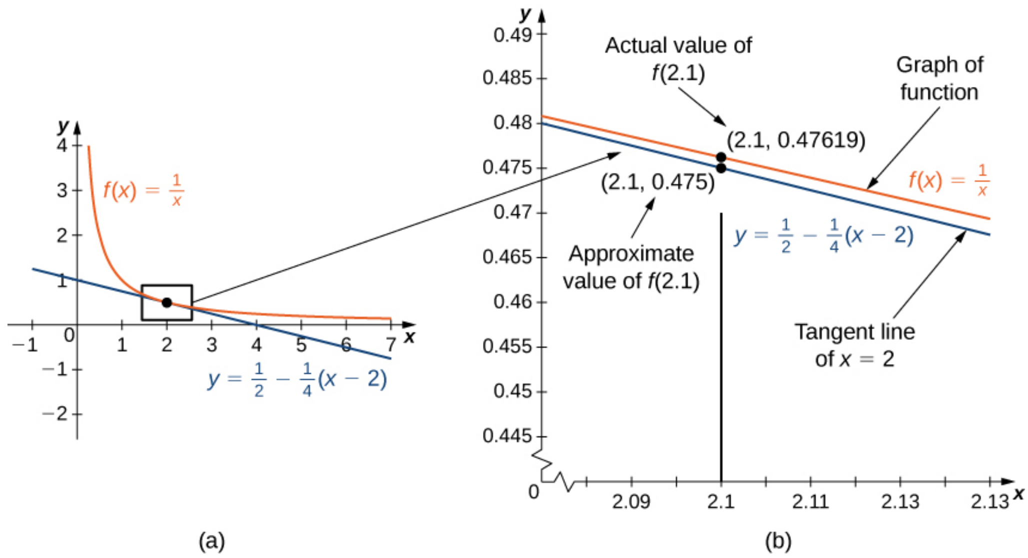


Figure 4.7 (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1,$ the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1,$ which is approximately 0.47619.

In general, for a differentiable function $f,$ the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near $a.$ Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \quad (4.1)$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a.$ This function L is also known as the **linearization** of f at $x = a.$

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9.$

Example 4.5

Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}.$

Solution

Since we are looking for the linear approximation at $x = 9,$ using **Equation 4.1** we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9).$

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(9) = \sqrt{9} = 3$$
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by **Figure 4.8**.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

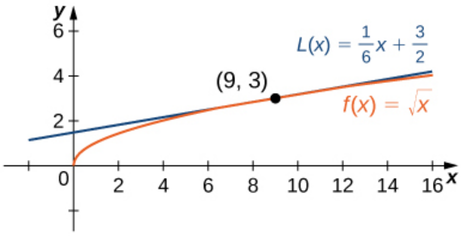


Figure 4.8 The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

4.5 Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Example 4.6

Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to $60°$, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

We see that

$$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 4.9.

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

To estimate $\sin(62°)$ using $L,$ we must first convert $62°$ to radians. We have $62° = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62°)$ is given by

$$\sin(62°) = f\left(\frac{62\pi}{180}\right) \approx L\left(\frac{62\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{62\pi}{180} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.88348.$$

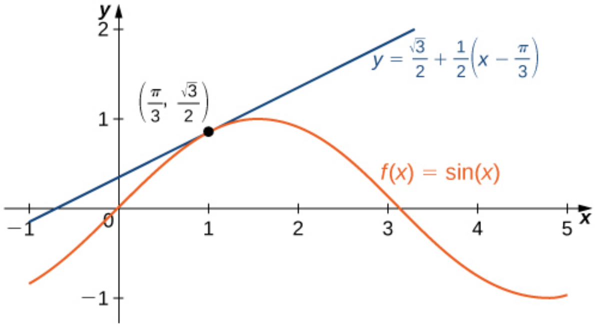


Figure 4.9 The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

4.6 Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0,$ which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

Example 4.7

Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0.$ Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$\begin{aligned} f(x) = (1 + x)^n \quad &\Rightarrow \quad f(0) = 1 \\ f'(x) = n(1 + x)^{n-1} \quad &\Rightarrow \quad f'(0) = n, \end{aligned}$$

the linear approximation is given by **Figure 4.10**(a).

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3.$ We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

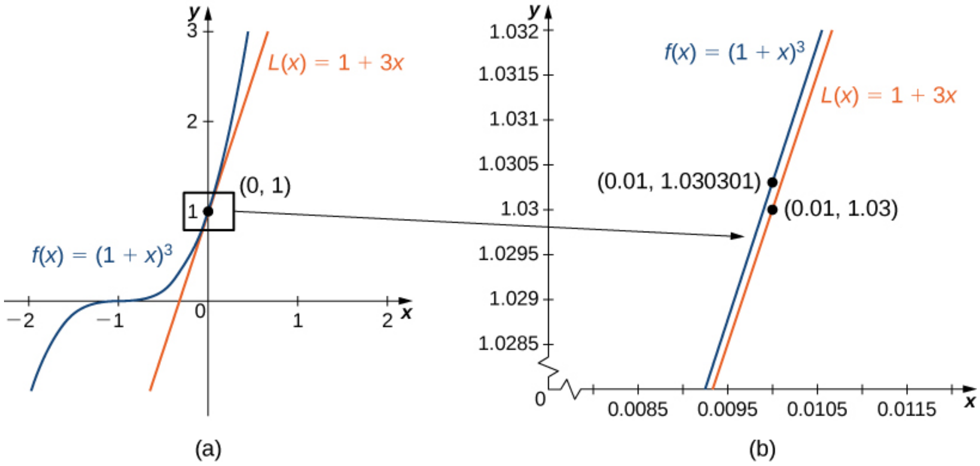


Figure 4.10 (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x).$ (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

4.7 Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to $x.$ Although we used the expressions *dy* and *dx* in this notation, they did not have meaning on their own. Here we see a meaning to the expressions *dy* and *dx*. Suppose $y = f(x)$ is a differentiable function. Let *dx* be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)dx. \tag{4.2}$$

It is important to notice that dy is a function of both x and $dx.$ The expressions *dy* and *dx* are called *differentials*. We can divide both sides of **Equation 4.2** by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{4.3}$$

This is the familiar expression we have used to denote a derivative. **Equation 4.2** is known as the **differential form** of **Equation 4.3**.

Example 4.8

Computing differentials

For each of the following functions, find *dy* and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$

b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain *dy* directly.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1 \sin(3).$$

4.8 For $y = e^{x^2}$, find dy.

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a+dx)-f(a)\approx L(a+dx)-f(a)=f'(a)dx.$$

In other words, the actual change in the function f if x increases from a to $a+dx$ is approximately the difference between $L(a+dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)dx$. In summary,

$$\Delta y=f(a+dx)-f(a)\approx L(a+dx)-f(a)=f'(a)dx=dy.$$

Therefore, we can use the differential $dy=f'(a)dx$ to approximate the change in y if x increases from $x=a$ to $x=a+dx$. We can see this in the following graph.

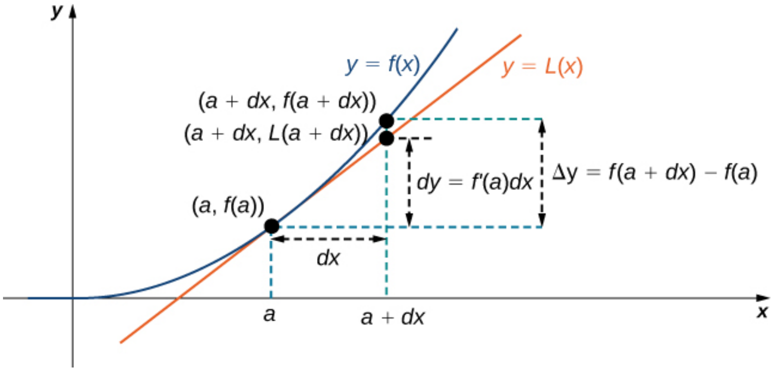


Figure 4.11 The differential $dy=f'(a)dx$ is used to approximate the actual change in y if x increases from a to $a+dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 4.9

Approximating Change with Differentials

Let $y=x^2+2x$. Compute Δy and dy at $x=3$ if $dx=0.1$.

Solution

The actual change in y if x changes from $x=3$ to $x=3.1$ is given by

$$\Delta y=f(3.1)-f(3)=[(3.1)^2+2(3.1)]-[3^2+2(3)]=0.81.$$

The approximate change in y is given by $dy=f'(3)dx$. Since $f'(x)=2x+2$, we have

$$dy=f'(3)dx=(2(3)+2)(0.1)=0.8.$$

4.9 For $y=x^2+2x$, find Δy and dy at $x=3$ if $dx=0.2$.

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a, the propagated error is

$$\Delta y \approx dy = f'(a)dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 4.10

Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \le dx \le 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \le dV \le 3(5)^2(0.1).$$

Therefore,

$$-7.5 \le dV \le 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649 \text{ cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651\ \text{cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125.$ Therefore, the error in the computed volume is

$$117.649 - 125 \le \Delta V \le 132.651 - 125.$$

That is,

$$-7.351 \le \Delta V \le 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

4.10 Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

The measurement error dx $(=\Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q},$ where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016,$ or $1.6\%.$ By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32},$ or $3.1\%.$ Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 4.11

Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution

If the measurement of the radius is accurate to within $\pm 80,$ we have

$$-80 \le dr \le 80.$$

Since the volume of a sphere is given by $V = \left(\frac{4}{3}\right)\pi r^3,$ we have

$$dV = 4\pi r^2\,dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \le dV \le 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx \left(\frac{4}{3}\right)\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \le \frac{dV}{V} \le \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \le \frac{dV}{V} \le 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

4.11 Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

4.2 EXERCISES

46. What is the linear approximation for any generic linear function $y = mx + b$?

47. Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

48. Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

49. When is the linear approximation exact?

For the following exercises, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

50. $f(x) = x + x^4, \ a = 0$

51. $f(x) = \frac{1}{x}, \ a = 2$

52. $f(x) = \tan x, \ a = \frac{\pi}{4}$

53. $f(x) = \sin x, \ a = \frac{\pi}{2}$

54. $f(x) = x \sin x, \ a = 2\pi$

55. $f(x) = \sin^2 x, \ a = 0$

For the following exercises, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

56. **[T]** $(2.001)^6$

57. **[T]** $\sin(0.02)$

58. **[T]** $\cos(0.03)$

59. **[T]** $(15.99)^{1/4}$

60. **[T]** $\frac{1}{0.98}$

61. **[T]** $\sin(3.14)$

For the following exercises, determine the appropriate $f(x)$ and a, and evaluate $L(x) = f(a) + f'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

62. **[T]** $(1.01)^3$

63. **[T]** $\cos(0.01)$

64. **[T]** $(\sin(0.01))^2$

65. **[T]** $(1.01)^{-3}$

66. **[T]** $\left(1 + \frac{1}{10}\right)^{10}$

67. **[T]** $\sqrt{8.99}$

For the following exercises, find the differential of the function.

68. $y = 3x^4 + x^2 - 2x + 1$

69. $y = x \cos x$

70. $y = \sqrt{1 + x}$

71. $y = \frac{x^2 + 2}{x - 1}$

For the following exercises, find the differential and evaluate for the given x and dx.

72. $y = 3x^2 - x + 6, \quad x = 2, \quad dx = 0.1$

73. $y = \frac{1}{x + 1}, \quad x = 1, \quad dx = 0.25$

74. $y = \tan x, \quad x = 0, \quad dx = \frac{\pi}{10}$

75. $y = \frac{3x^2 + 2}{\sqrt{x + 1}}, \quad x = 0, \quad dx = 0.1$

76. $y = \frac{\sin(2x)}{x}, \quad x = \pi, \quad dx = 0.25$

77. $y = x^3 + 2x + \frac{1}{x}, \quad x = 1, \quad dx = 0.05$

For the following exercises, find the change in volume dV or in surface area dA.

78. dV if the sides of a cube change from 10 to 10.1.

79. dA if the sides of a cube change from x to $x + dx$.

80. dA if the radius of a sphere changes from r by dr.

81. dV if the radius of a sphere changes from r by dr.

82. dV if a circular cylinder with $r = 2$ changes height from 3 cm to $3.05\,\text{cm}$.

83. dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9\,\text{cm}$.

For the following exercises, use differentials to estimate the maximum and relative error when computing the surface area or volume.

84. A spherical golf ball is measured to have a radius of $5\,\text{mm},$ with a possible measurement error of $0.1\,\text{mm}.$ What is the possible change in volume?

85. A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

86. An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

For the following exercises, confirm the approximations by using the linear approximation at $x = 0.$

87. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$

88. $\frac{1}{\sqrt{1 - x^2}} \approx 1$

89. $\sqrt{c^2 + x^2} \approx c$

4.3 | Maxima and Minima

Learning Objectives

4.3.1 Define absolute extrema.
4.3.2 Define local extrema.
4.3.3 Explain how to find the critical points of a function over a closed interval.
4.3.4 Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (see the following figure).

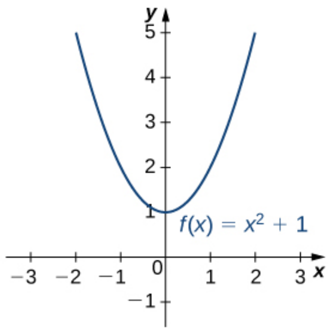


Figure 4.12 The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition

Let f be a function defined over an interval I and let $c \in I$. We say f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an **absolute extremum** on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is

$f(0) = 1.$ It occurs at $x = 0,$ as shown in **Figure 4.13**(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. **Figure 4.13** shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

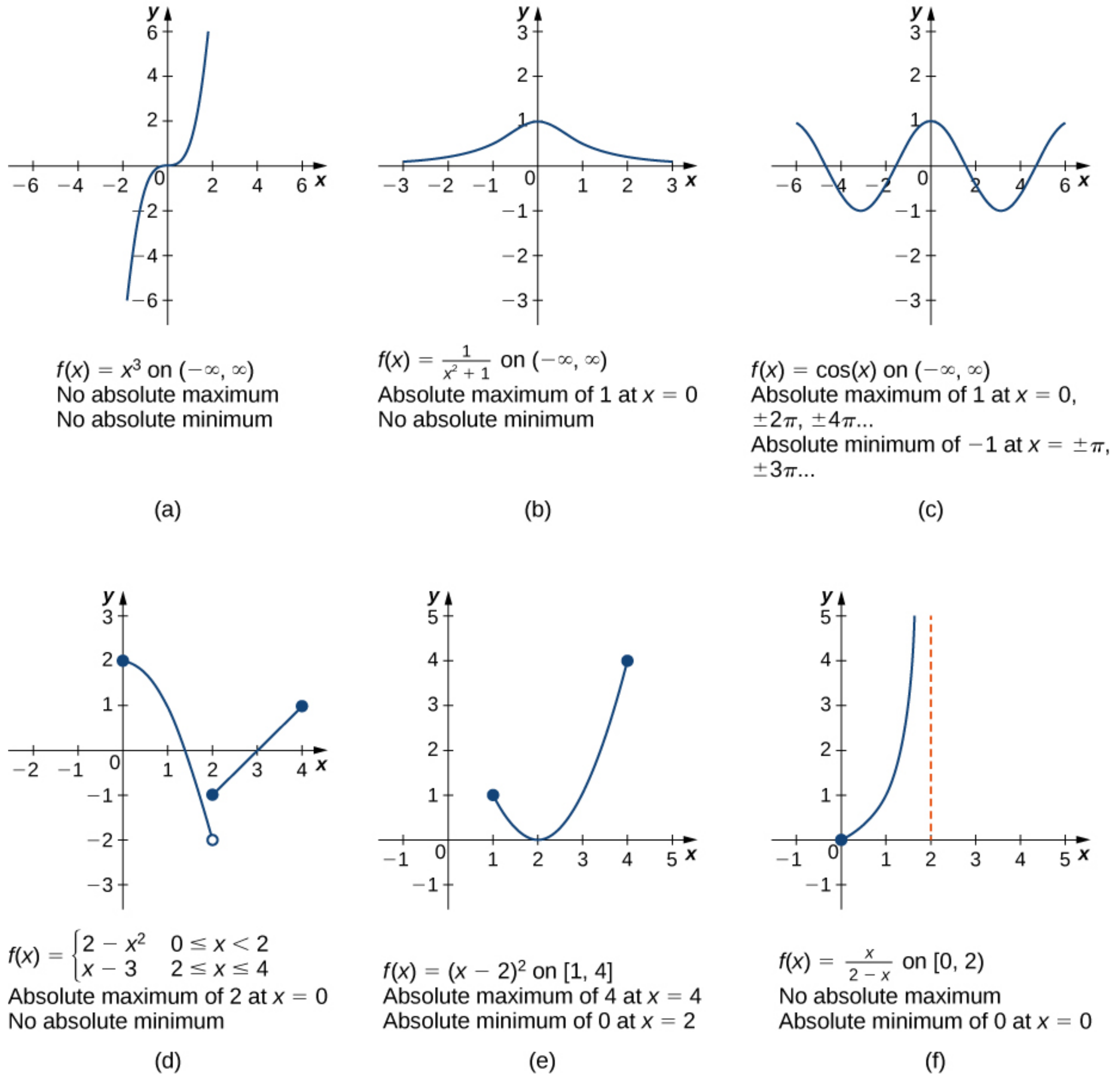


Figure 4.13 Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty).$ Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 4.1: Extreme Value Theorem

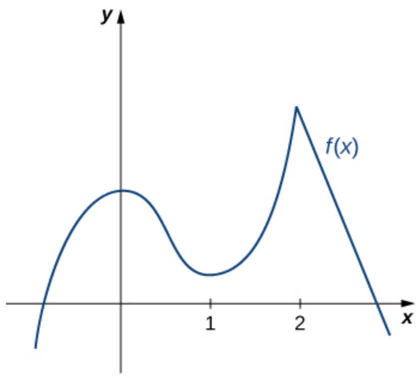
If f is a continuous function over the closed, bounded interval $[a, b],$ then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b].$

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 4.13(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure 4.14. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 4.14 This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition

A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at **Figure 4.14** again. The local extrema occur at $x = 0,$ $x = 1,$ and $x = 2.$ Notice that at $x = 0$ and $x = 1,$ the derivative $f'(x) = 0.$ At $x = 2,$ the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c,$ the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition

Let c be an interior point in the domain of f. We say that c is a **critical point** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c,$ then c must be a critical point of f. This fact is known as **Fermat's theorem**.

Theorem 4.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at $c,$ then $f'(c) = 0.$

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0.$ To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0,$ and therefore $f'(c) = 0.$ Since f has a local extremum at $c,$ f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I.$ Since f is differentiable at $c,$ from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c).$ Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}, \tag{4.4}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}. \tag{4.5}$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near $c,$ but $x > c,$ we have $\frac{f(x) - f(c)}{x - c} \leq 0.$ From **Equation 4.4** we conclude that $f'(c) \leq 0.$ Similarly, it can be shown that $f'(c) \geq 0.$ Therefore, $f'(c) = 0.$

□

From Fermat's theorem, we conclude that if f has a local extremum at $c,$ then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In **Figure 4.15**, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

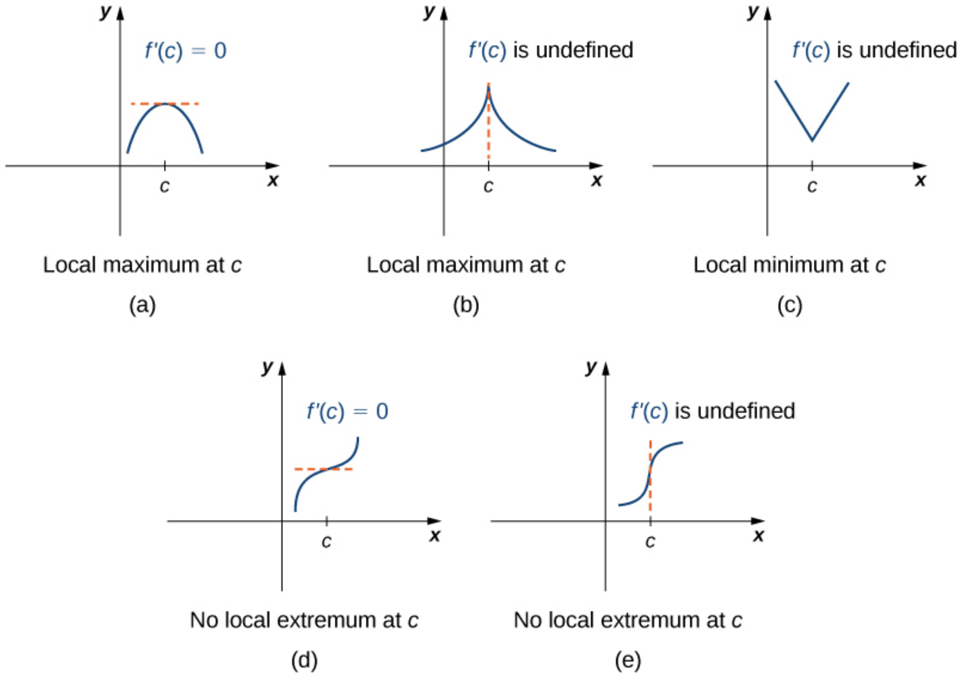


Figure 4.15 (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.12

Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $f(x) = \left(x^2 - 1\right)^3$

c. $f(x) = \dfrac{4x}{1 + x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in **Figure 4.16**, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

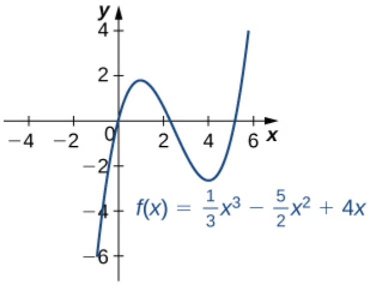


Figure 4.16 This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3\left(x^2 - 1\right)^2 (2x) = 6x\left(x^2 - 1\right)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in **Figure 4.17**, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

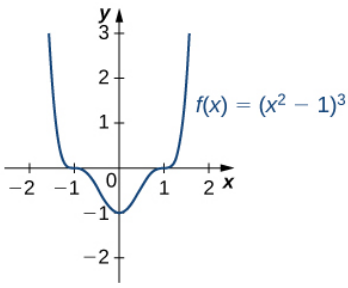


Figure 4.17 This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is

$$f'(x) = \frac{\left(1 + x^2 4\right) - 4x(2x)}{\left(1 + x^2\right)^2} = \frac{4 - 4x^2}{\left(1 + x^2\right)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in **Figure 4.18**, we see that f has an absolute maximum at $x = 1$

and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

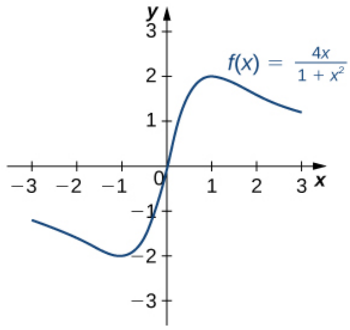


Figure 4.18 This function has an absolute maximum and an absolute minimum.

4.12 Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in **Figure 4.13**, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by **Fermat's Theorem**, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From **Location of Absolute Extrema**, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.13

Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.

b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f\left(\frac{3}{2}\right)$ and find

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	**Conclusion**
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval [1, 3] is -2, and it occurs at $x = 3$ as shown in the following graph.

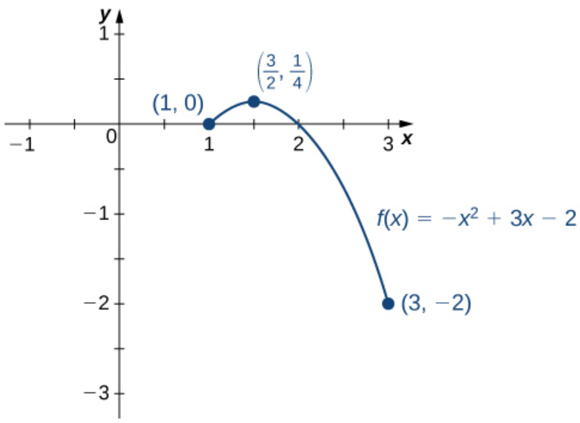


Figure 4.19 This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\sqrt[3]{4} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	**Conclusion**
0	0	Absolute maximum
1	−2	Absolute minimum
2	−0.762	

We conclude that the absolute maximum of f over the interval [0, 2] is zero, and it occurs at $x = 0$. The absolute minimum is −2, and it occurs at $x = 1$ as shown in the following graph.

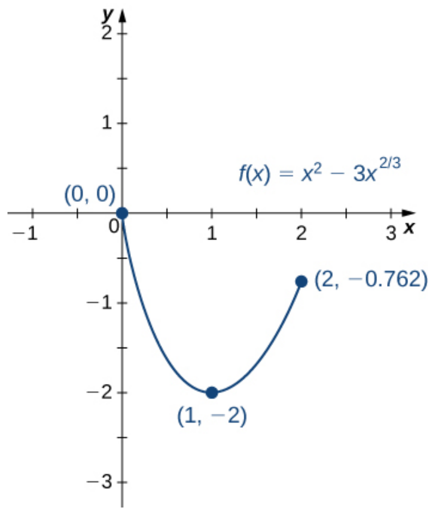


Figure 4.20 This function has an absolute maximum at an endpoint of the interval.

4.13 Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

4.3 EXERCISES

90. In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $h = -\frac{b}{(2a)}$. Prove this formula using calculus.

91. If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

92. If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

93. When you are checking for critical points, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

94. Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

95. Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

96. Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

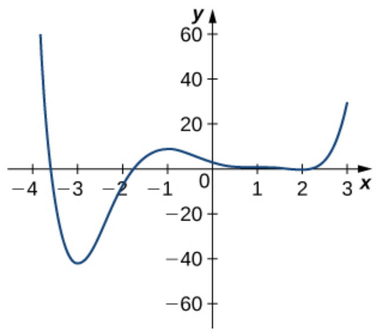
97. Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

98. Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

99. **[T]** Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

For the following exercises, determine where the local and absolute maxima and minima occur on the graph given. Assume the graph represents the entirety of each function.

100.



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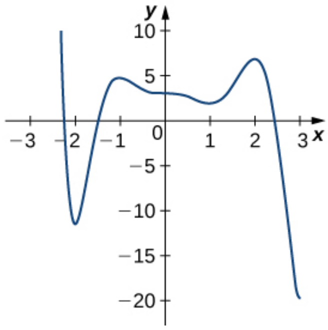
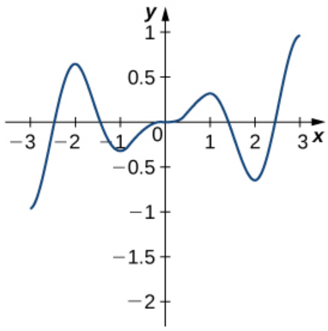
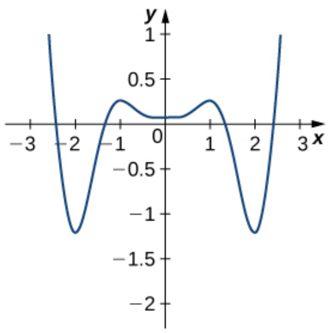


102.



103.



For the following problems, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

104. Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

105. Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

106. Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

107. Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

For the following exercises, find the critical points in the domains of the following functions.

108. $y = 4x^3 - 3x$

109. $y = 4\sqrt{x} - x^2$

110. $y = \frac{1}{x-1}$

111. $y = \ln(x-2)$

112. $y = \tan(x)$

113. $y = \sqrt{4 - x^2}$

114. $y = x^{3/2} - 3x^{5/2}$

115. $y = \frac{x^2 - 1}{x^2 + 2x - 3}$

116. $y = \sin^2(x)$

117. $y = x + \frac{1}{x}$

For the following exercises, find the local and/or absolute maxima for the functions over the specified domain.

118. $f(x) = x^2 + 3$ over $[-1, 4]$

119. $y = x^2 + \frac{2}{x}$ over $[1, 4]$

120. $y = \left(x - x^2\right)^2$ over $[-1, 1]$

121. $y = \frac{1}{\left(x - x^2\right)}$ over $(0, 1)$

122. $y = \sqrt{9 - x}$ over $[1, 9]$

123. $y = x + \sin(x)$ over $[0, 2\pi]$

124. $y = \frac{x}{1 + x}$ over $[0, 100]$

125. $y = |x + 1| + |x - 1|$ over $[-3, 2]$

126. $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

127. $y = \sin x + \cos x$ over $[0, 2\pi]$

128. $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

129. $y = x^2 + 4x + 5$

130. $y = x^3 - 12x$

131. $y = 3x^4 + 8x^3 - 18x^2$

132. $y = x^3(1 - x)^6$

133. $y = \frac{x^2 + x + 6}{x - 1}$

134. $y = \frac{x^2 - 1}{x - 1}$

For the following functions, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

135. **[T]** $y = 3x\sqrt{1 - x^2}$

136. **[T]** $y = x + \sin(x)$

137. **[T]** $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

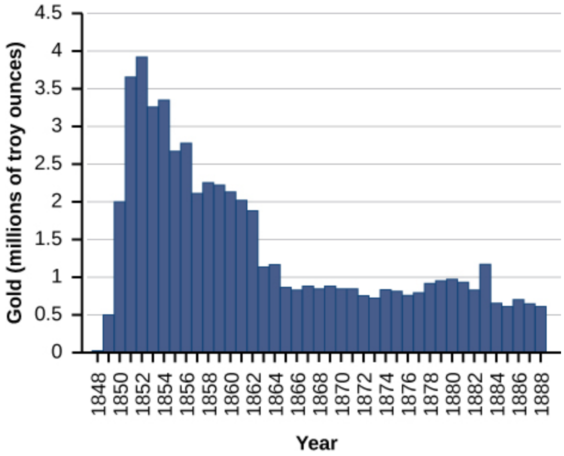
138. **[T]** $y = \frac{x^3 + 6x^2 - x - 30}{x - 2}$

139. **[T]** $y = \frac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}$

140. A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36{,}400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

141. A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5\text{ m}$. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

For the following exercises, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{\left(t^2 + 16\right)}$, where t is the number of years since the rush began $(0 \le t \le 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



142. Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

143. Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Find the critical points, maxima, and minima for the following piecewise functions.

144. $y = \begin{cases} x^2 - 4x & 0 \le x \le 1 \\ x^2 - 4 & 1 < x \le 2 \end{cases}$

145. $y = \begin{cases} x^2 + 1 & x \le 1 \\ x^2 - 4x + 5 & x > 1 \end{cases}$

For the following exercises, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

146. $y = ax^2 + bx + c$, given that $a > 0$

147. $y = (x - 1)^a$, given that $a > 1$ and a is an integer.

4.4 | The Mean Value Theorem

Learning Objectives

4.4.1 Explain the meaning of Rolle's theorem.
4.4.2 Describe the significance of the Mean Value Theorem.
4.4.3 State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure 4.21 illustrates this theorem.

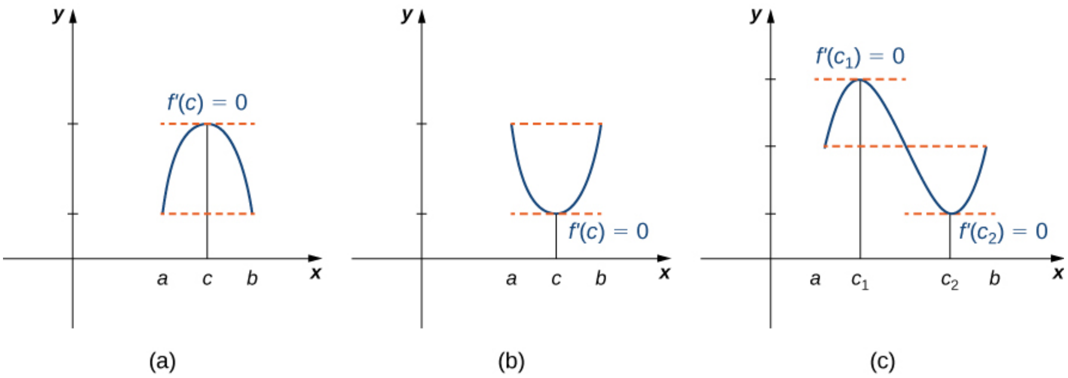


Figure 4.21 If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Theorem 4.4: Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1),$ but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

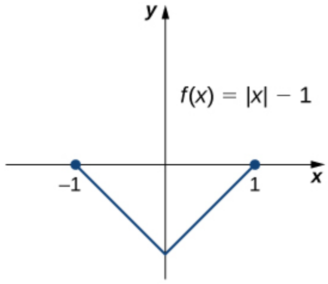


Figure 4.22 Since $f(x) = |x| - 1$ is not differentiable at $x = 0,$ the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0.$

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0.$

Example 4.14

Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0.$

a. $f(x) = x^2 + 2x$ over $[-2, 0]$

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0).$ Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0.$ Since $f'(x) = 2x + 2 = 2(x + 1),$ we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

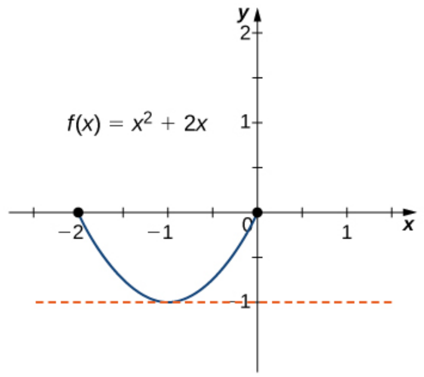


Figure 4.23 This function is continuous and differentiable over $[-2, 0]$, $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

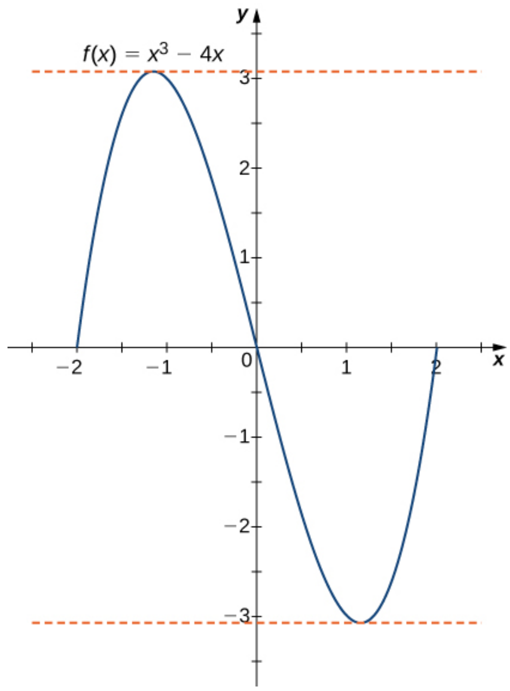


Figure 4.24 For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

4.14 Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (**Figure 4.25**). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

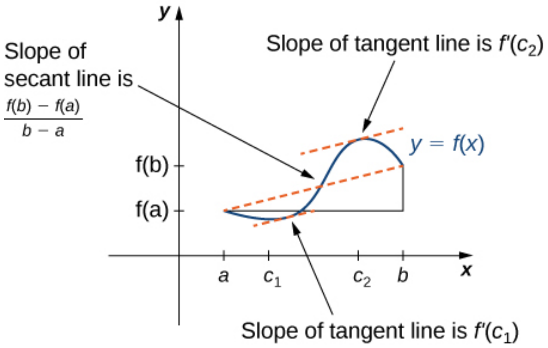


Figure 4.25 The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Theorem 4.5: Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

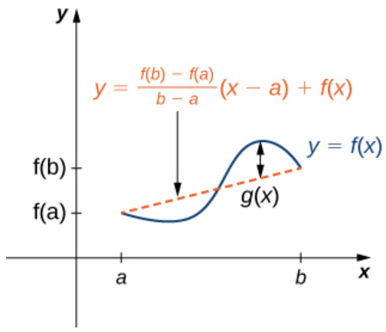


Figure 4.26 The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.15

Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (**Figure 4.27**). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

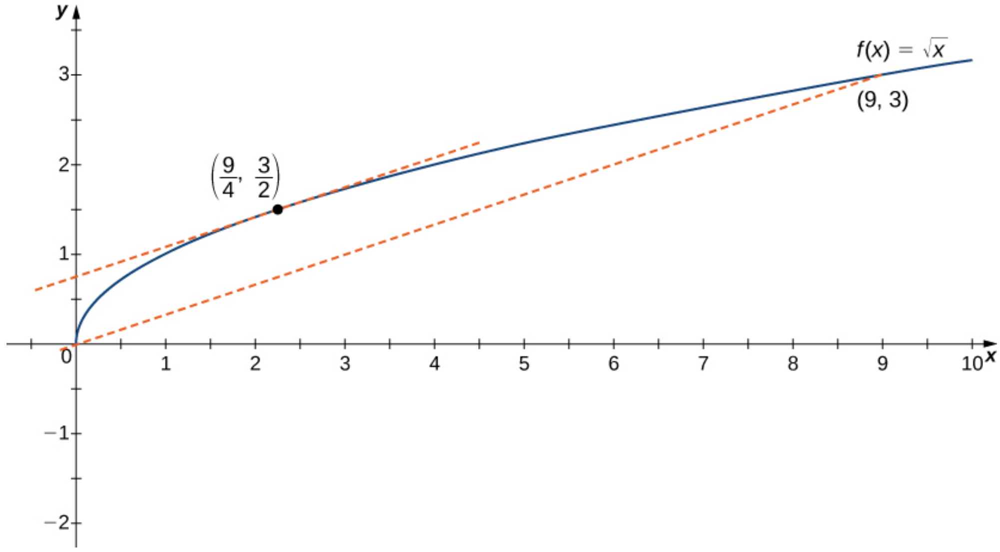


Figure 4.27 The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting $(0, 0)$ and $(9, 3)$.

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.16

Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.

b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.

c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2}$sec. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec.}$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{avg} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

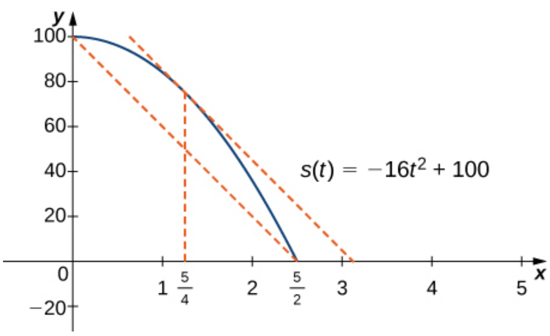


Figure 4.28 At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

4.15 Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Theorem 4.6: Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From **Corollary 1: Functions with a Derivative of Zero**, it follows that if two functions have the same derivative, they differ by, at most, a constant.

Theorem 4.7: Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x)_1 > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (**Figure 4.29**). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

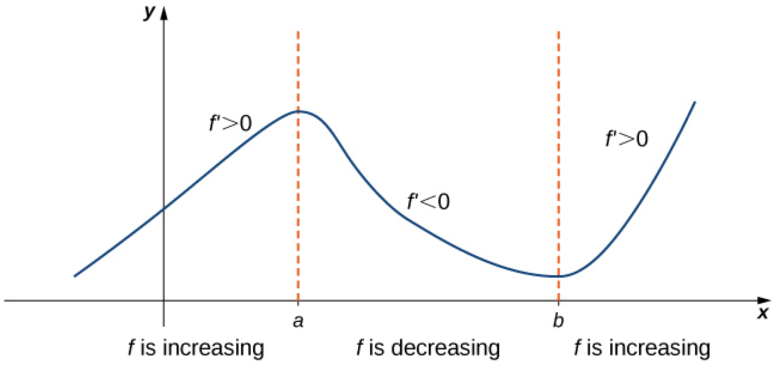


Figure 4.29 If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Theorem 4.8: Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.

ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

4.4 EXERCISES

148. Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

149. Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

150. When are Rolle's theorem and the Mean Value Theorem equivalent?

151. If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a)=f(b)-f(a)$? Draw such an example or prove why not.

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

152. $y=\sin(\pi x)$

153. $y=\frac{1}{x^3}$

154. $y=\sqrt{4-x^2}$

155. $y=\sqrt{x^2-4}$

156. $y=\ln(3x-5)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a)=f(b)-f(a)$.

157. **[T]** $y=3x^3+2x+1$ over $[-1, 1]$

158. **[T]** $y=\tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

159. **[T]** $y=x^2\cos(\pi x)$ over $[-2, 2]$

160. **[T]** $y=x^6-\frac{3}{4}x^5-\frac{9}{8}x^4+\frac{15}{16}x^3+\frac{3}{32}x^2+\frac{3}{16}x+\frac{1}{32}$ over $[-1, 1]$

For the following exercises, use the Mean Value Theorem and find all points $0<c<2$ such that $f(2)-f(0)=f'(c)(2-0)$.

161. $f(x)=x^3$

162. $f(x)=\sin(\pi x)$

163. $f(x)=\cos(2\pi x)$

164. $f(x)=1+x+x^2$

165. $f(x)=(x-1)^{10}$

166. $f(x)=(x-1)^9$

For the following exercises, show there is no c such that $f(1)-f(-1)=f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

167. $f(x)=\left|x-\frac{1}{2}\right|$

168. $f(x)=\frac{1}{x^2}$

169. $f(x)=\sqrt{|x|}$

170. $f(x)=\lfloor x\rfloor$ (*Hint*: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

171. $y=e^x$ over $[0, 1]$

172. $y=\ln(2x+3)$ over $\left[-\frac{3}{2}, 0\right]$

173. $f(x)=\tan(2\pi x)$ over $[0, 2]$

174. $y=\sqrt{9-x^2}$ over $[-3, 3]$

175. $y=\frac{1}{|x+1|}$ over $[0, 3]$

176. $y=x^3+2x+1$ over $[0, 6]$

177. $y=\frac{x^2+3x+2}{x}$ over $[-1, 1]$

178. $y=\frac{x}{\sin(\pi x)+1}$ over $[0, 1]$

179. $y=\ln(x+1)$ over $[0, e-1]$

180. $y=x\sin(\pi x)$ over $[0, 2]$

181. $y = 5 + |x|$ over $[-1, 1]$

For the following exercises, consider the roots of the equation.

182. Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

183. Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

184. Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

For the following exercises, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

185. **[T]** $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

186. **[T]** $y = \frac{1}{\sqrt{x+1}}$ over $[0, 3]$

187. **[T]** $y = \left|x^2 + 2x - 4\right|$ over $[-4, 0]$

188. **[T]** $y = x + \frac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

189. **[T]** $y = \sqrt{x+1} + \frac{1}{x^2}$ over $[3, 8]$

190. At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

191. Two cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

192. Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

193. Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

4.5 | Derivatives and the Shape of a Graph

Learning Objectives

4.5.1 Explain how the sign of the first derivative affects the shape of a function's graph.
4.5.2 State the first derivative test for critical points.
4.5.3 Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
4.5.4 Explain the concavity test for a function over an open interval.
4.5.5 Explain the relationship between a function and its first and second derivatives.
4.5.6 State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point $c,$ then c must be a critical point of $f.$ However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0,$ but f does not have a local extremum at $x = 0.$ Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over $I.$ On the other hand, if the derivative of the function is negative over an interval $I,$ then the function is decreasing over I as shown in the following figure.

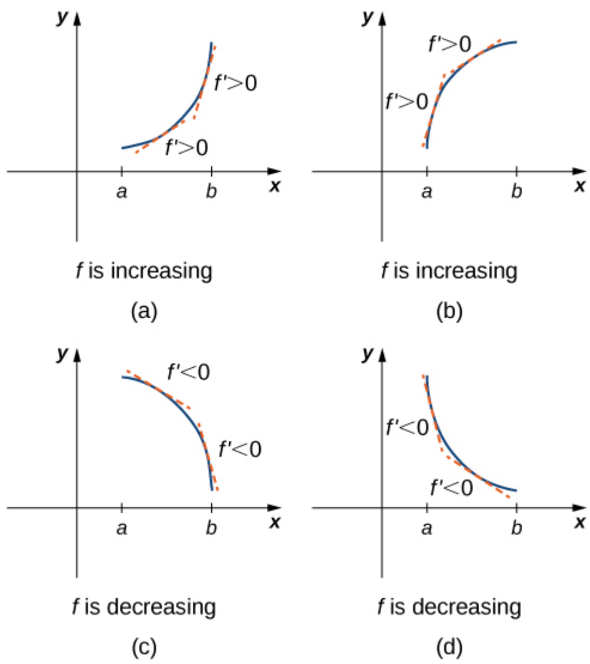


Figure 4.30 Both functions are increasing over the interval $(a, b).$ At each point $x,$ the derivative $f'(x) > 0.$ Both functions are decreasing over the interval $(a, b).$ At each point $x,$ the derivative $f'(x) < 0.$

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f'. can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In **Figure 4.31**, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

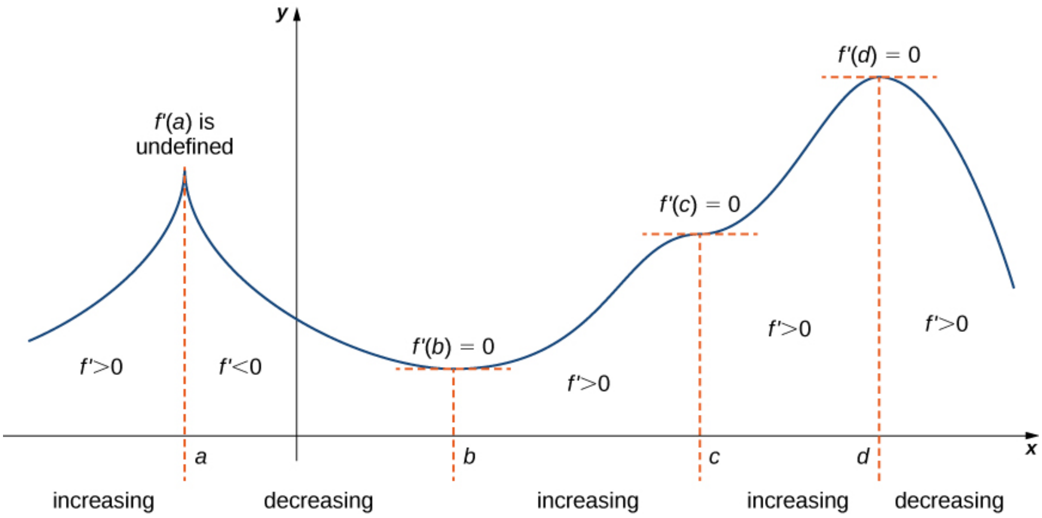


Figure 4.31 The function f has four critical points: a, b, c, and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using **Figure 4.31**, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test**.

Theorem 4.9: First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.

ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.

iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

We can summarize the first derivative test as a strategy for locating local extrema.

Problem-Solving Strategy: Using the First Derivative Test

Consider a function f that is continuous over an interval I.

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of f' in each of the subintervals. If f' is continuous over a given subinterval (which is typically the case), then the sign of f' in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of f' at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.
3. Use **First Derivative Test** and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.17

Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2,$ $x = 0,$ and $x = 4$ as test points.

Interval	**Test Point**	**Sign of $f'(x) = 3(x - 3)(x + 1)$ at Test Point**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 3)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through $1,$ f has a local maximum at $x = -1.$ Since f' switches sign from negative to positive as x increases through $3,$ f has a local minimum at $x = 3.$ These analytical results agree with the following graph.

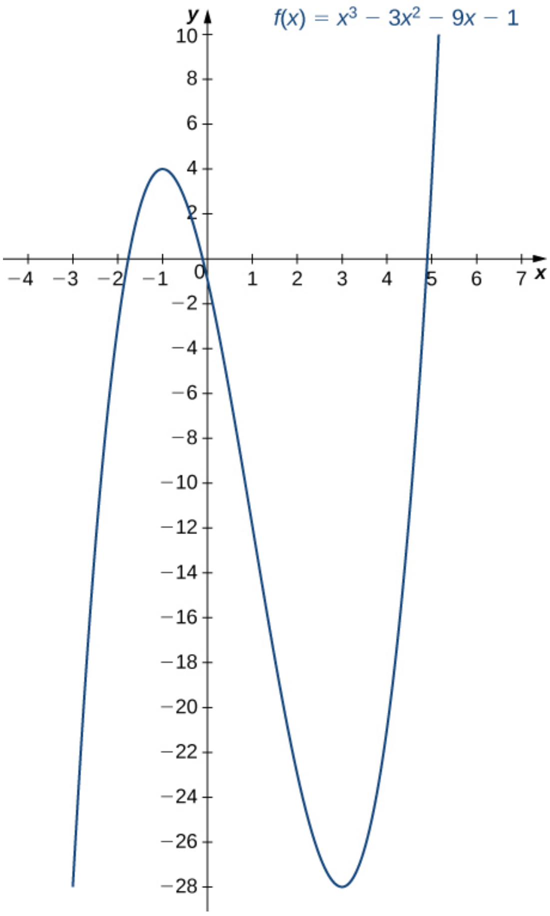


Figure 4.32 The function f has a maximum at $x = -1$ and a minimum at $x = 3$

4.16 Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x.$

Example 4.18

Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}.$ Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5\left(1 - x^{4/3}\right)}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0.$ Therefore, $f'(x) = 0$ at $x = \pm 1.$ The derivative $f'(x)$ is undefined at $x = 0.$ Therefore, we have three critical points: $x = 0,$ $x = 1,$ and $x = -1.$ Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 0), (0, 1),$ and $(1, \infty).$

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2, x = -\frac{1}{2}, x = \frac{1}{2},$ and $x = 2$ are test points for these intervals.

Interval	**Test Point**	**Sign of $f'(x) = \frac{5\left(1 - x^{4/3}\right)}{3x^{2/3}}$ at Test Point**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0),$ f has a local minimum at $x = -1.$ Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1),$ f does not have a local extremum at $x = 0.$ Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty),$ f has a local maximum at $x = 1.$ The analytical results agree with the following graph.

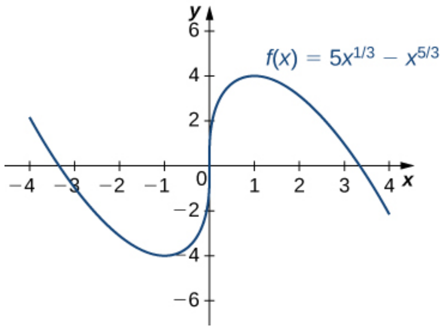


Figure 4.33 The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$.

4.17 Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x-1}$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.34(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. **Figure 4.34**(b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

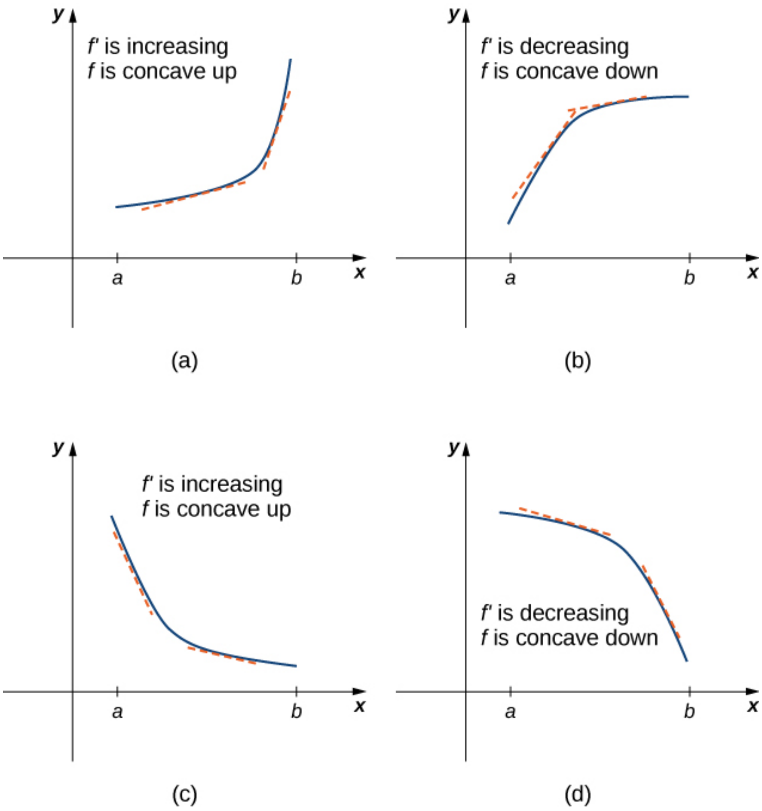


Figure 4.34 (a), (c) Since f' is increasing over the interval $(a, b),$ we say f is concave up over $(a, b).$ (b), (d) Since f' is decreasing over the interval $(a, b),$ we say f is concave down over $(a, b).$

In general, without having the graph of a function $f,$ how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary $3,$ we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0.$ Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0.$ Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0.$ Therefore, a twice-differentiable function f is concave down when $f''(x) < 0.$ Applying this logic is known as the **concavity test**.

Theorem 4.10: Test for Concavity

Let f be a function that is twice differentiable over an interval $I.$

i. If $f''(x) > 0$ for all $x \in I,$ then f is concave up over $I.$

ii. If $f''(x) < 0$ for all $x \in I,$ then f is concave down over $I.$

We conclude that we can determine the concavity of a function f by looking at the second derivative of $f.$ In addition, we observe that a function f can switch concavity (Figure 4.35). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point $x,$ then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at $a,$ we say the point $(a, f(a))$ is an inflection point of f.

Definition

If f is continuous at a and f changes concavity at $a,$ the point $(a, f(a))$ is an **inflection point** of f.

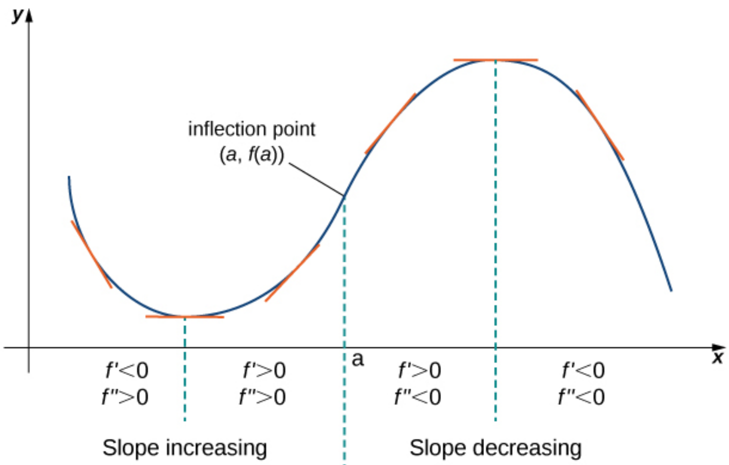


Figure 4.35 Since $f''(x) > 0$ for $x < a,$ the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a,$ the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.19

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9,$ so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers $x,$ we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0,$ we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. **Figure 4.36** confirms the analytical results.

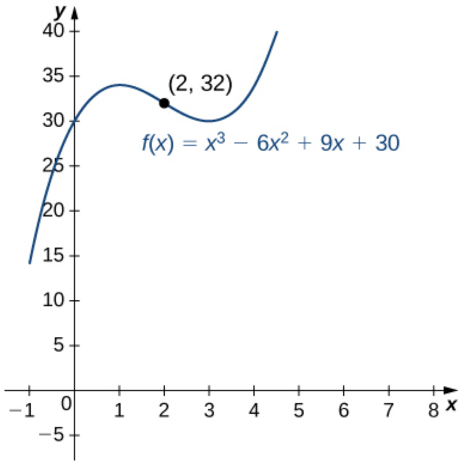


Figure 4.36 The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

4.18 For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

We now summarize, in **Table 4.1**, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in **Figure 4.37**.

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.1 What Derivatives Tell Us about Graphs

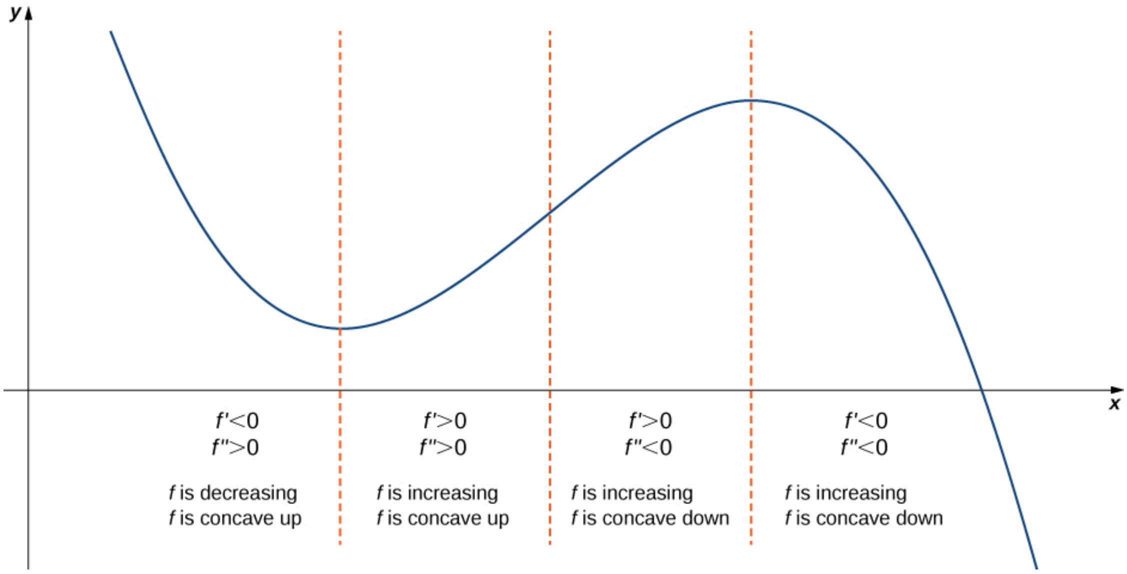


Figure 4.37 Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (**Figure 4.38**). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (**Figure 4.38**). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

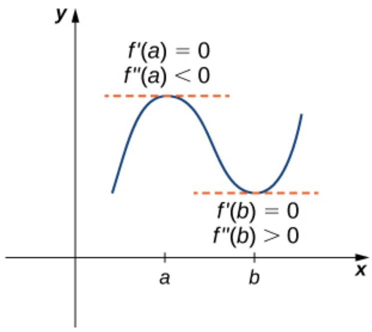


Figure 4.38 Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0,$ there is an interval I containing a such that for all x in $I,$ f is increasing if $x < a$ and f is decreasing if $x > a.$ As a result, f has a local maximum at $x = a.$ Since $f'(b) = 0$ and $f''(b) > 0,$ there is an interval I containing b such that for all x in $I,$ f is decreasing if $x < b$ and f is increasing if $x > b.$ As a result, f has a local minimum at $x = b.$

Theorem 4.11: Second Derivative Test

Suppose $f'(c) = 0,$ f'' is continuous over an interval containing $c.$

i. If $f''(c) > 0,$ then f has a local minimum at $c.$

ii. If $f''(c) < 0,$ then f has a local maximum at $c.$

iii. If $f''(c) = 0,$ then the test is inconclusive.

Note that for case iii. when $f''(c) = 0,$ then f may have a local maximum, local minimum, or neither at $c.$ For example, the functions $f(x) = x^3,$ $f(x) = x^4,$ and $f(x) = -x^4$ all have critical points at $x = 0.$ In each case, the second derivative is zero at $x = 0.$ However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x,$ and the function $f(x) = x^3$ does not have a local extremum at $x = 0.$

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0.$

Example 4.20

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3.$

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0.$ The derivative is

$f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2\left(x^2 - 3\right)$ for $x \in \left(-\sqrt{3}, 0\right)$ and $x \in \left(0, \sqrt{3}\right)$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

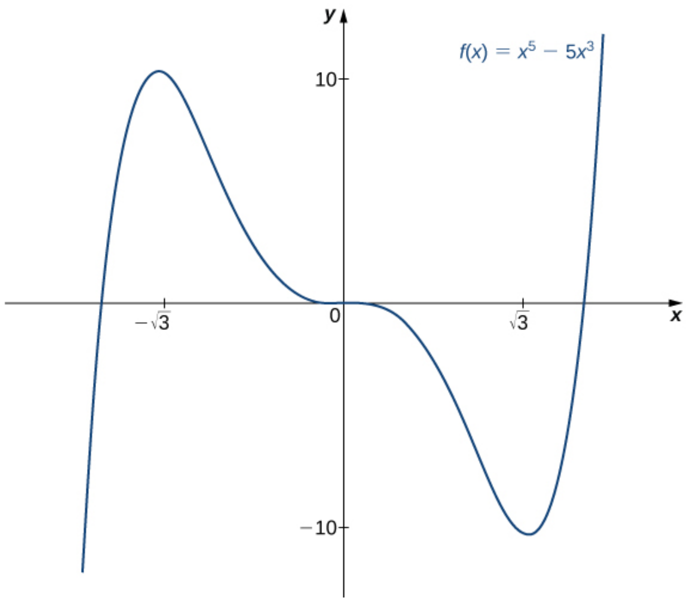


Figure 4.39 The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

4.19 Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

4.5 EXERCISES

194. If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

195. For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

196. For the function $y = x^3$, is $x = 0$ an inflection point?

197. Is it possible for a point c to be both an inflection point and a local extrema of a twice differentiable function?

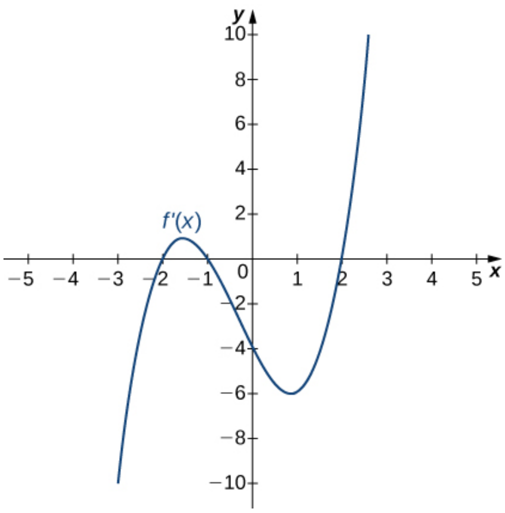
198. Why do you need continuity for the first derivative test? Come up with an example.

199. Explain whether a concave-down function has to cross $y = 0$ for some value of x.

200. Explain whether a polynomial of degree 2 can have an inflection point.

For the following exercises, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

201.



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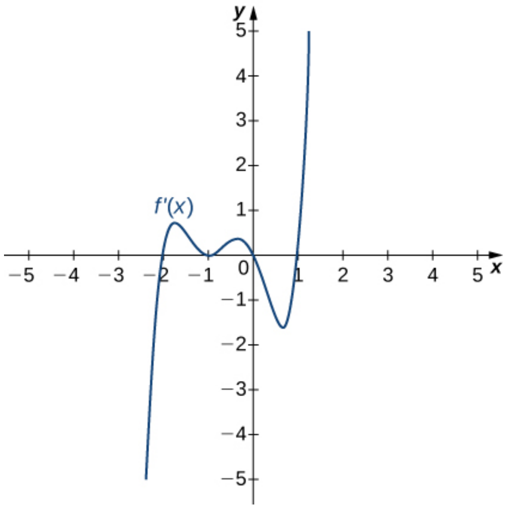

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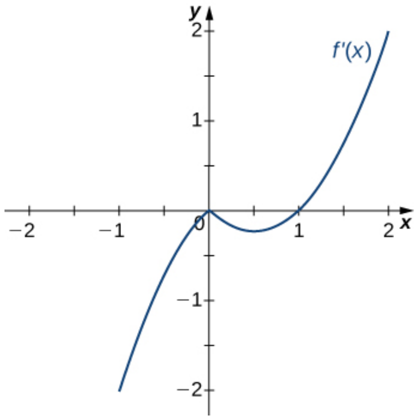

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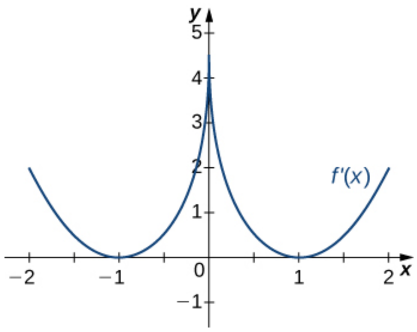

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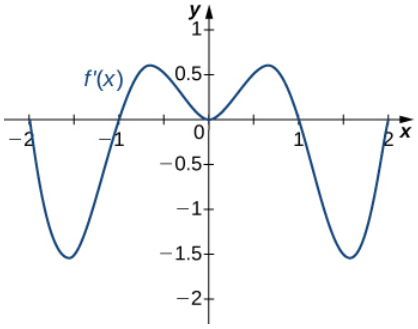
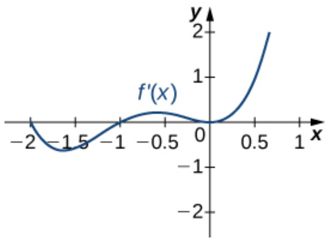


For the following exercises, analyze the graphs of f', then list all intervals where

a. f is increasing and decreasing and

b. the minima and maxima are located.

206.



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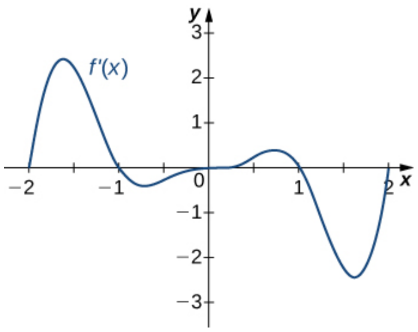

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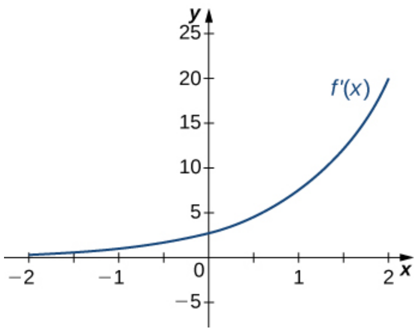

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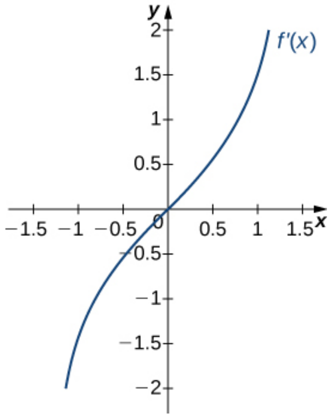

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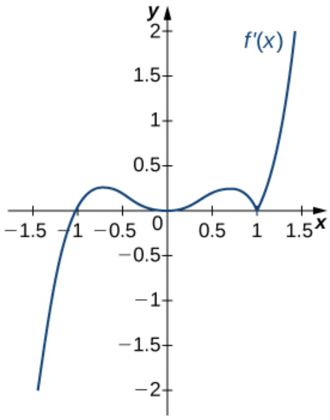
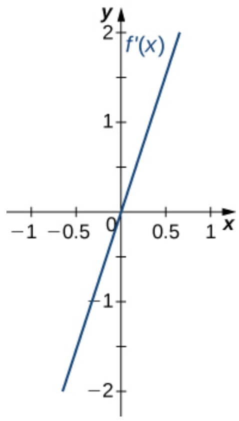


For the following exercises, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

211.



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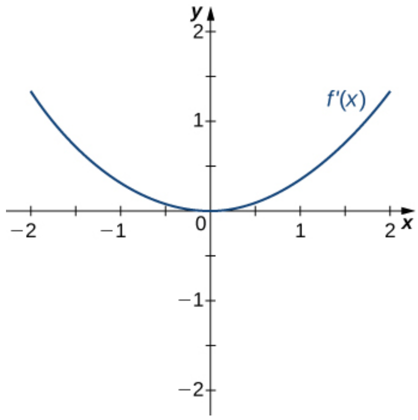

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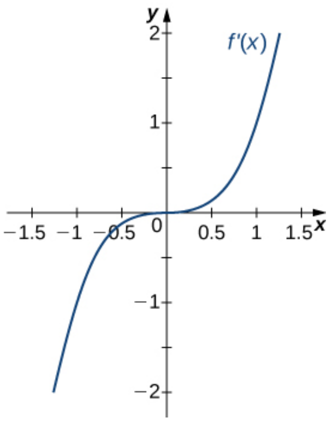

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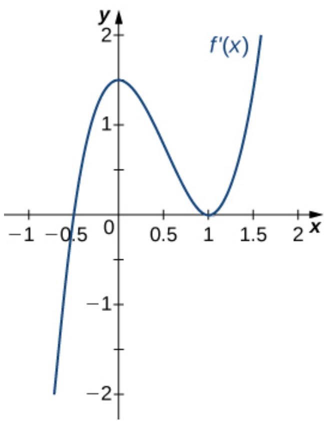

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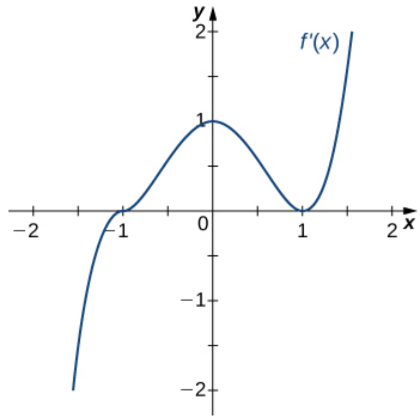


For the following exercises, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

216. $f(x) > 0, f'(x) > 0$ over $x > 1, -3 < x < 0, f'(x) = 0$ over $0 < x < 1$

217. $f'(x) > 0$ over $x > 2, -3 < x < -1, f'(x) < 0$ over $-1 < x < 2, f''(x) < 0$ for all x

218. $f''(x) < 0$ over $-1 < x < 1, f''(x) > 0, -3 < x < -1, 1 < x < 3,$ local maximum at $x = 0,$ local minima at $x = \pm 2$

219. There is a local maximum at $x = 2,$ local minimum at $x = 1,$ and the graph is neither concave up nor concave down.

220. There are local maxima at $x = \pm 1,$ the function is concave up for all $x,$ and the function remains positive for all $x.$

For the following exercises, determine

a. intervals where f is increasing or decreasing and
b. local minima and maxima of f.

221. $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

222. $f(x) = x^2 + \cos x$

For the following exercises, determine a. intervals where f is concave up or concave down, and b. the inflection points of f.

223. $f(x) = x^3 - 4x^2 + x + 2$

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

224. $f(x) = x^2 - 6x$

225. $f(x) = x^3 - 6x^2$

226. $f(x) = x^4 - 6x^3$

227. $f(x) = x^{11} - 6x^{10}$

228. $f(x) = x + x^2 - x^3$

229. $f(x) = x^2 + x + 1$

230. $f(x) = x^3 + x^4$

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

231. **[T]** $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

232. **[T]** $f(x) = x + \sin(2x)$ over $x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

233. **[T]** $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

234. **[T]** $f(x) = (x - 2)^2 (x - 4)^2$

235. **[T]** $f(x) = \frac{1}{1 - x}, \ x \neq 1$

236. **[T]** $f(x) = \frac{\sin x}{x}$ over $x =$ $[2\pi, 0) \cup (0, 2\pi]$

237. $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

238. $f(x) = \ln x \sqrt{x}, \ x > 0$

239. $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, \ x > 0$

240. $f(x) = \frac{e^x}{x}, \ x \neq 0$

For the following exercises, interpret the sentences in terms of f, f', and f''.

241. The population is growing more slowly. Here f is the population.

242. A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

243. The airplane lands smoothly. Here f is the plane's altitude.

244. Stock prices are at their peak. Here f is the stock price.

245. The economy is picking up speed. Here f is a measure of the economy, such as GDP.

For the following exercises, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0, f'(3) = 0$. Determine whether the following statements are *true or false*. Justify your answer.

246. $f(x) = 0$ for some $1 \leq x \leq 3$

247. $f''(x) = 0$ for some $1 \leq x \leq 3$

248. There is no absolute maximum at $x = 3$

249. If $f(x)$ has three roots, then it has 1 inflection point.

250. If $f(x)$ has one inflection point, then it has three real roots.

4.6 | Limits at Infinity and Asymptotes

Learning Objectives

4.6.1 Calculate the limit of a function as x increases or decreases without bound.
4.6.2 Recognize a horizontal asymptote on the graph of a function.
4.6.3 Estimate the end behavior of a function as x increases or decreases without bound.
4.6.4 Recognize an oblique asymptote on the graph of a function.
4.6.5 Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

Limits at Infinity

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in **Introduction to Functions and Graphs**, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in **Figure 4.40** and numerically in **Table 4.2**, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim\limits_{x \to \infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approaches 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim\limits_{x \to -\infty} f(x) = 2$.

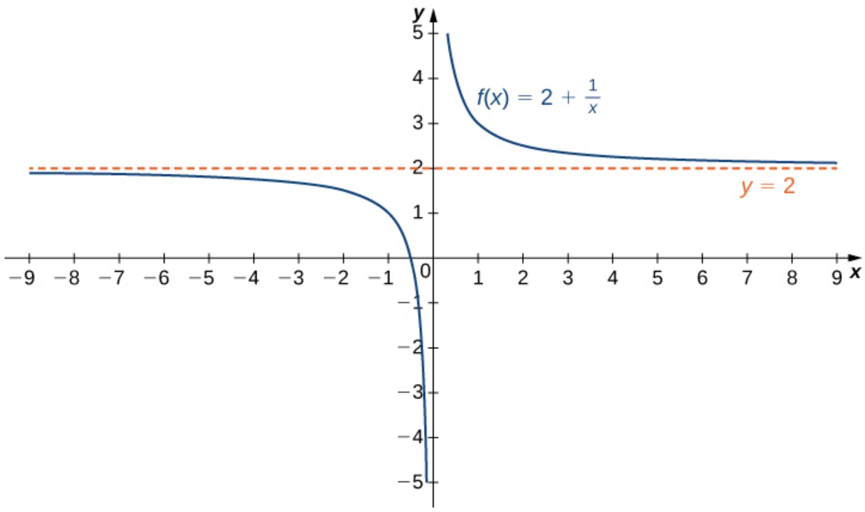


Figure 4.40 The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

x	10	100	1,000	10,000
$2+\frac{1}{x}$	2.1	2.01	2.001	2.0001
x	-10	-100	-1000	$-10{,}000$
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

Table 4.2 Values of a function f as $x \to \pm\infty$

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim_{x \to \infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x \to -\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition

(Informal) If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x \to \infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x \to -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (**Figure 4.41**). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

Definition

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

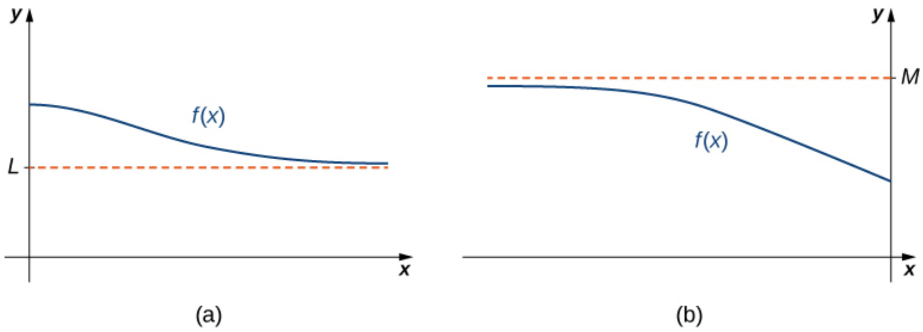


Figure 4.41 (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{(\cos x)}{x} + 1$ shown in **Figure 4.42** intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

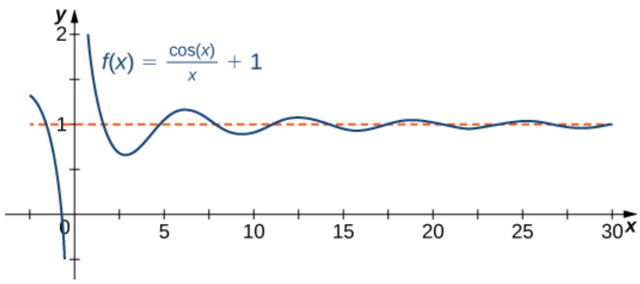


Figure 4.42 The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in **Introduction to Limits** also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 4.21

Computing Limits at Infinity

For each of the following functions f, evaluate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $f(x) = 5 - \frac{2}{x^2}$

b. $f(x) = \frac{\sin x}{x}$

c. $f(x) = \tan^{-1}(x)$

Solution

a. Using the algebraic limit laws, we have $\lim_{x \to \infty}\left(5 - \frac{2}{x^2}\right) = \lim_{x \to \infty} 5 - 2\left(\lim_{x \to \infty}\frac{1}{x}\right)\cdot\left(\lim_{x \to \infty}\frac{1}{x}\right) = 5 - 2 \cdot 0 = 5.$

Similarly, $\lim_{x \to -\infty} f(x) = 5.$ Therefore, $f(x) = 5 - \frac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

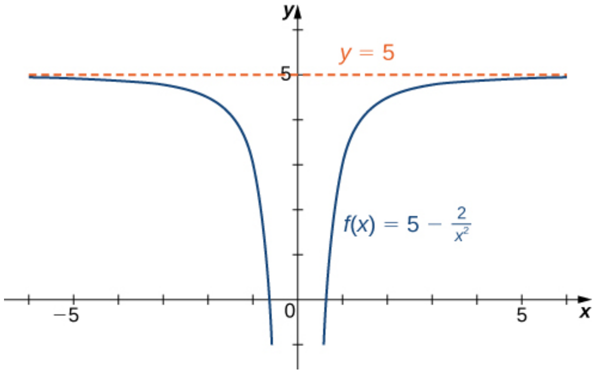


Figure 4.43 This function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since $-1 \le \sin x \le 1$ for all x, we have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

for all $x \ne 0$. Also, since

$$\lim_{x \to \infty}\frac{-1}{x} = 0 = \lim_{x \to \infty}\frac{1}{x},$$

we can apply the squeeze theorem to conclude that

$$\lim_{x \to \infty}\frac{\sin x}{x} = 0.$$

Similarly,

$$\lim_{x \to -\infty}\frac{\sin x}{x} = 0.$$

Thus, $f(x) = \frac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

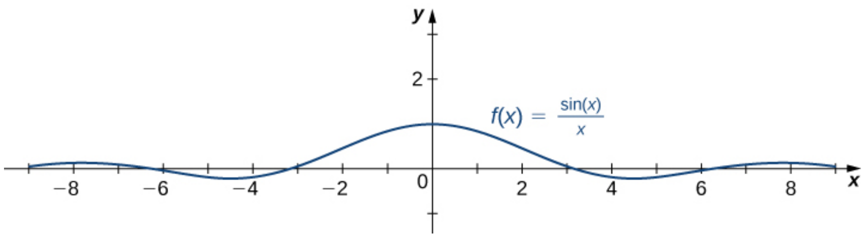


Figure 4.44 This function crosses its horizontal asymptote multiple times.

c. To evaluate $\lim_{x \to \infty} \tan^{-1}(x)$ and $\lim_{x \to -\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\pi/2, \pi/2)$ as shown in the following graph.

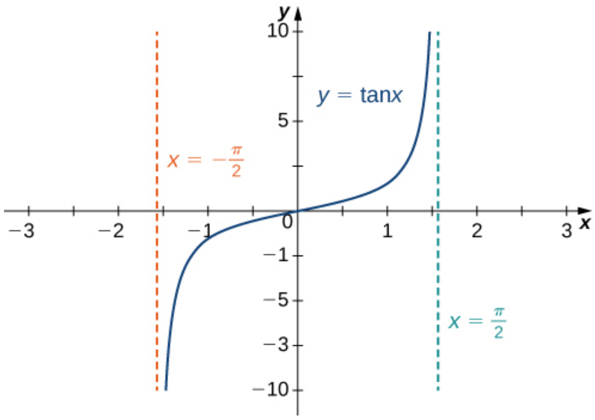


Figure 4.45 The graph of $\tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$

Since

$$\lim_{x \to (\pi/2)^-} \tan x = \infty,$$

it follows that

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

$$\lim_{x \to (-\pi/2)^+} \tan x = -\infty,$$

it follows that

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.

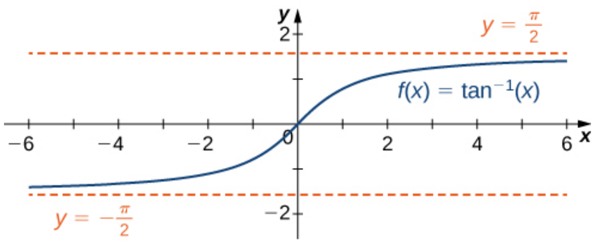


Figure 4.46 This function has two horizontal asymptotes.

4.20 Evaluate $\lim_{x \to -\infty} \left(3 + \frac{4}{x}\right)$ and $\lim_{x \to \infty} \left(3 + \frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim_{x \to \infty} f(x) = \infty$ (or $\lim_{x \to -\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x \to \infty} f(x) = -\infty$ (or $\lim_{x \to -\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in **Table 4.3** and **Figure 4.47**, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x \to \infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x \to -\infty} x^3 = -\infty$.

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

Table 4.3 Values of a power function as $x \to \pm\infty$

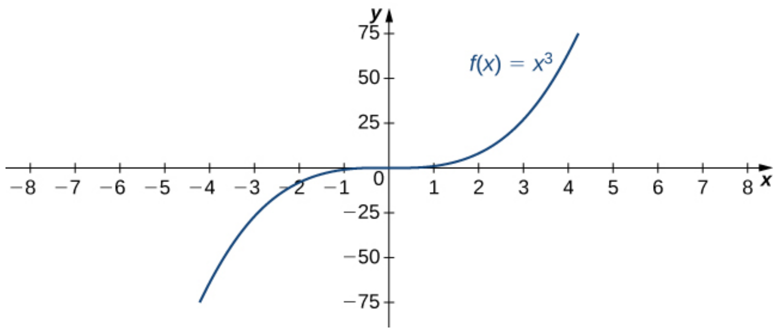


Figure 4.47 For this function, the functional values approach infinity as $x \to \pm\infty$.

Definition

(Informal) We say a function f has an infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

Formal Definitions

Earlier, we used the terms *arbitrarily close*, *arbitrarily large*, and *sufficiently large* to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Definition

(Formal) We say a function f has a **limit at infinity**, if there exists a real number L such that for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N$. In that case, we write

$$\lim_{x \to \infty} f(x) = L$$

(see **Figure 4.48**).

We say a function f has a limit at negative infinity if there exists a real number L such that for all $\varepsilon > 0$, there exists $N < 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x < N$. In that case, we write

$$\lim_{x \to -\infty} f(x) = L.$$

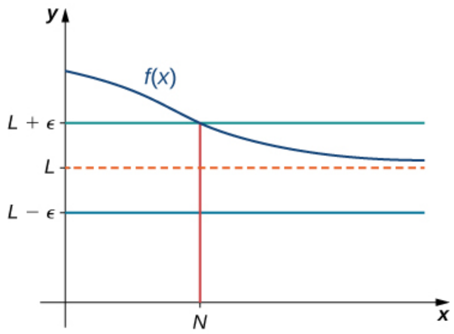


Figure 4.48 For a function with a limit at infinity, for all $x > N$, $|f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in **Figure 4.40** and numerical evidence in **Table 4.2** to conclude that $\lim_{x \to \infty}\left(2 + \frac{1}{x}\right) = 2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example 4.22 A Finite Limit at Infinity Example

Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty}\left(2 + \frac{1}{x}\right) = 2$.

Solution

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

$$\left|2 + \frac{1}{x} - 2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \varepsilon.$$

4.21 Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty}\left(3 - \frac{1}{x^2}\right) = 3$.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Definition

(Formal) We say a function f has an **infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see **Figure 4.49**).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

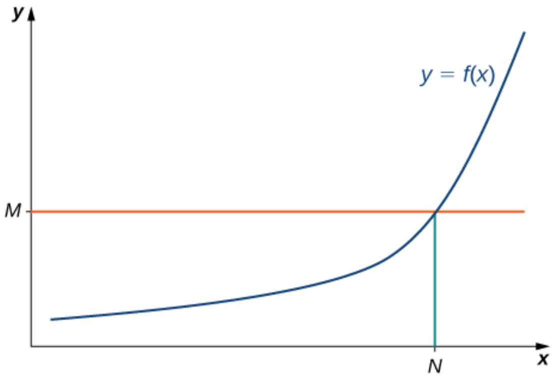


Figure 4.49 For a function with an infinite limit at infinity, for all $x > N$, $f(x) > M$.

Earlier, we used graphical evidence (**Figure 4.47**) and numerical evidence (**Table 4.3**) to conclude that $\lim\limits_{x \to \infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example 4.23 An Infinite Limit at Infinity

Use the formal definition of infinite limit at infinity to prove that $\lim\limits_{x \to \infty} x^3 = \infty$.

Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > N^3 = \left(\sqrt[3]{M}\right)^3 = M.$$

Therefore, $\lim\limits_{x \to \infty} x^3 = \infty$.

4.22 Use the formal definition of infinite limit at infinity to prove that $\lim\limits_{x \to \infty} 3x^2 = \infty$.

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From **Figure 4.50** and **Figure 4.51**, we see that

$$\lim_{x \to \infty} x^n = \infty;\ n = 1, 2, 3,\ldots$$

and

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty;\ n = 2, 4, 6,\ldots \\ -\infty;\ n = 1, 3, 5,\ldots \end{cases}.$$

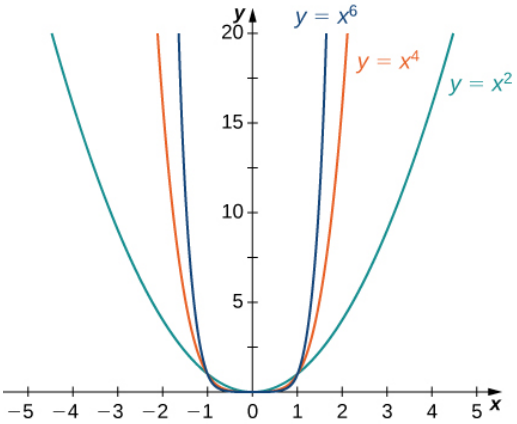


Figure 4.50 For power functions with an even power of n, $\lim_{x \to \infty} x^n = \infty = \lim_{x \to -\infty} x^n$.

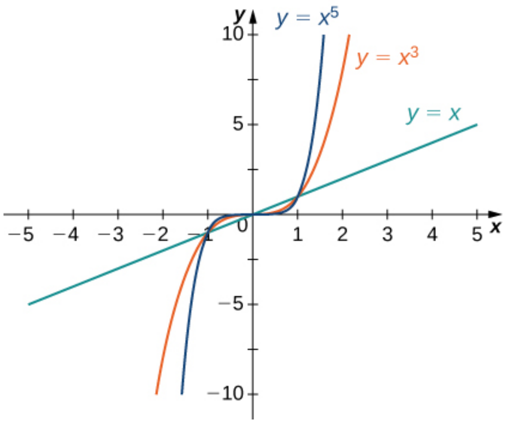


Figure 4.51 For power functions with an odd power of n, $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x \to \infty} cx^n$ and $\lim_{x \to -\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x \to \infty} cx^n = \lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = \lim_{x \to -\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x \to \infty} cx^n = -\lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = -\lim_{x \to -\infty} x^n \text{ if } c < 0.$$

If $c = 0,\ y = cx^n = 0,$ in which case $\lim_{x \to \infty} cx^n = 0 = \lim_{x \to -\infty} cx^n.$

Example 4.24

Limits at Infinity for Power Functions

For each function $f,$ evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x).$

a. $f(x) = -5x^3$

b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is $-5,$ the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim_{x \to \infty} \left(-5x^3\right) = -\infty$ and $\lim_{x \to -\infty} \left(-5x^3\right) = \infty.$

b. Since the coefficient of x^4 is $2,$ the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4.$ Therefore, $\lim_{x \to \infty} 2x^4 = \infty$ and $\lim_{x \to -\infty} 2x^4 = \infty.$

4.23 Let $f(x) = -3x^4.$ Find $\lim_{x \to \infty} f(x).$

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x \to \pm\infty} f(x)$ for any polynomial function $f.$ Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

of degree $n \geq 1$ so that $a_n \neq 0.$ Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty,$ all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in **Figure 4.52** and **Table 4.4**.

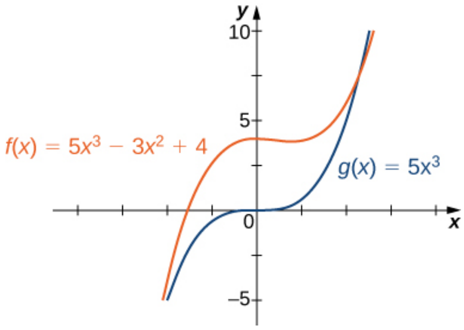


Figure 4.52 The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−1000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5,029,996	−5,002,999,996
$g(x) = 5x^3$	−5000	−5,000,000	−5,000,000,000

Table 4.4 A polynomial's end behavior is determined by the term with the largest exponent.

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In **Example 4.25**, we show that the limits at infinity of a rational function $f(x) = \frac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 4.25

Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \frac{3x - 1}{2x + 5}$ (*Note:* The degree of the numerator and the denominator are the same.)

b. $f(x) = \dfrac{3x^2 + 2x}{4x^3 - 5x + 7}$ (*Note:* The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ (*Note:* The degree of numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x \to \pm\infty} \frac{3x - 1}{2x + 5} &= \lim_{x \to \pm\infty} \frac{3 - 1/x}{2 + 5/x} \\ &= \frac{\lim_{x \to \pm\infty}(3 - 1/x)}{\lim_{x \to \pm\infty}(2 + 5/x)} \\ &= \frac{\lim_{x \to \pm\infty} 3 - \lim_{x \to \pm\infty} 1/x}{\lim_{x \to \pm\infty} 2 + \lim_{x \to \pm\infty} 5/x} \\ &= \frac{3 - 0}{2 + 0} = \frac{3}{2}.\end{aligned}$$

Since $\lim_{x \to \pm\infty} f(x) = \frac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

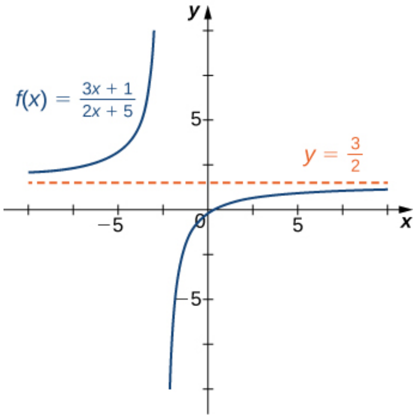


Figure 4.53 The graph of this rational function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x \to \pm\infty} \frac{3x^2 + 2x}{4x^3 - 5x + 7} = \lim_{x \to \pm\infty} \frac{3/x + 2/x^2}{4 - 5/x^2 + 7/x^3} = \frac{3(0) + 2(0)}{4 - 5(0) + 7(0)} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.

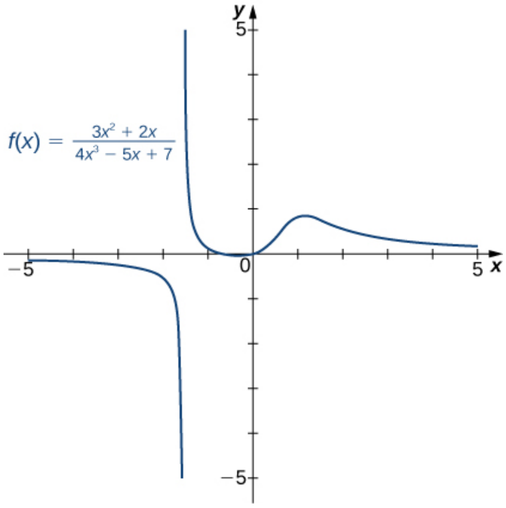


Figure 4.54 The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x \to \pm\infty} \frac{3x^2 + 4x}{x + 2} = \lim_{x \to \pm\infty} \frac{3x + 4}{1 + 2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x \to \infty} f(x) = \infty$, whereas $\lim_{x \to -\infty} f(x) = -\infty$ as shown in the following figure.

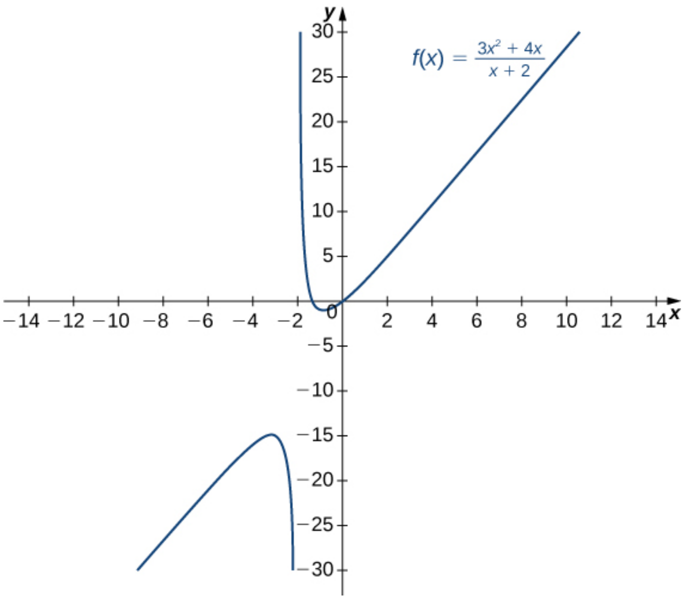


Figure 4.55 As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

4.24 Evaluate $\lim_{x \to \pm\infty} \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$.

Before proceeding, consider the graph of $f(x) = \frac{(3x^2 + 4x)}{(x + 2)}$ shown in **Figure 4.56**. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\frac{4}{(x + 2)} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty}(f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (**Figure 4.56**).

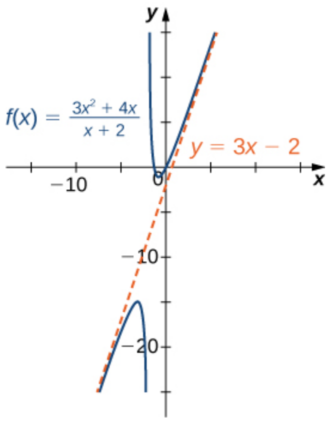


Figure 4.56 The graph of the rational function $f(x) = (3x^2 + 4x)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$.

We can summarize the results of **Example 4.25** to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator $(n = m)$, then f has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator $(n < m)$, then f has a horizontal asymptote of $y = 0$ as $x \to \pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator $(n > m)$, then f does not have a

horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim_{x \to \pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ $(n = m + 1)$, the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 4.26

Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{3x - 2}{\sqrt{4x^2 + 5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2 + 5}$ in the denominator. Since

$$\sqrt{4x^2 + 5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} &= \lim_{x \to \infty} \frac{(1/|x|)(3x - 2)}{(1/|x|)\sqrt{4x^2 + 5}} \\
&= \lim_{x \to \infty} \frac{(1/x)(3x - 2)}{\sqrt{\left(1/x^2\right)\left(4x^2 + 5\right)}} \\
&= \lim_{x \to \infty} \frac{3 - 2/x}{\sqrt{4 + 5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2} \\
\lim_{x \to -\infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} &= \lim_{x \to -\infty} \frac{(1/|x|)(3x - 2)}{(1/|x|)\sqrt{4x^2 + 5}} \\
&= \lim_{x \to -\infty} \frac{(-1/x)(3x - 2)}{\sqrt{\left(1/x^2\right)\left(4x^2 + 5\right)}} \\
&= \lim_{x \to -\infty} \frac{-3 + 2/x}{\sqrt{4 + 5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}.
\end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \to -\infty$ as shown in the following graph.

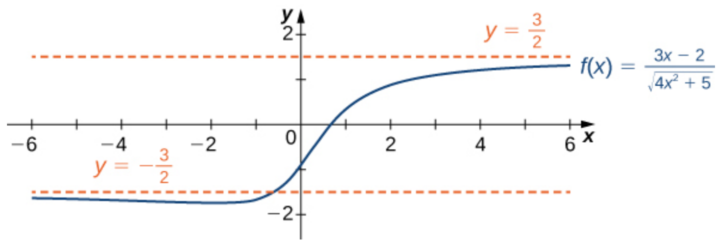


Figure 4.57 This function has two horizontal asymptotes and it crosses one of the asymptotes.

4.25 Evaluate $\lim\limits_{x \to \infty} \frac{\sqrt{3x^2+4}}{x+6}$.

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin x$ oscillates between $1 \text{ and } -1$ (**Figure 4.58**). The tangent function x has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in **Figure 4.59**.

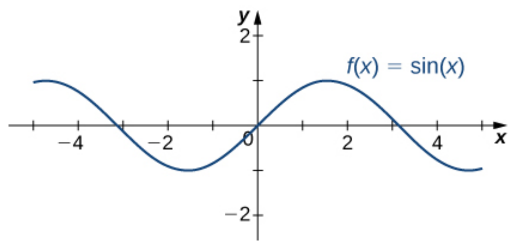


Figure 4.58 The function $f(x) = \sin x$ oscillates between $1 \text{ and } -1$ as $x \to \pm\infty$

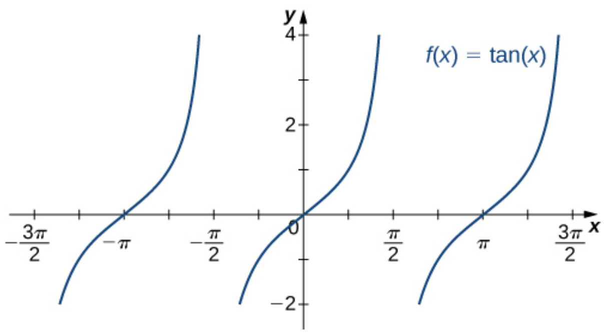


Figure 4.59 The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0, b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1$, $y = b^x$ is increasing over $(-\infty, \infty)$. If $0 < b < 1$, $y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x$, $e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in **Table 4.5** and **Figure 4.60**.

x	-5	-2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

Table 4.5 End behavior of the natural exponential function

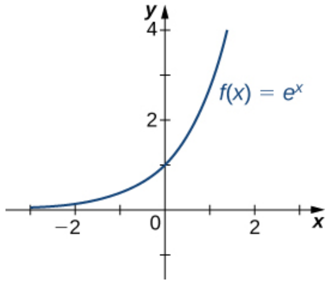


Figure 4.60 The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in **Figure 4.61** and **Table 4.6**.

x	0.01	0.1	1	10	100
$\ln(x)$	-4.605	-2.303	0	2.303	4.605

Table 4.6 End behavior of the natural logarithm function

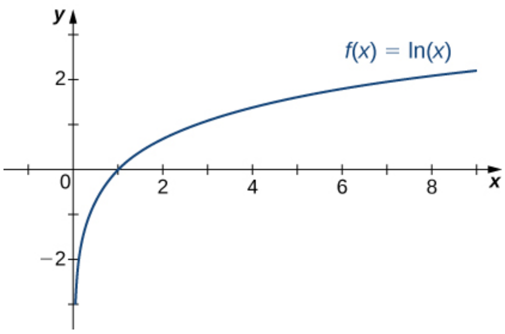


Figure 4.61 The natural logarithm function approaches ∞ as $x \to \infty$.

Example 4.27

Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{(2+3e^x)}{(7-5e^x)}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{2+3e^x}{7-5e^x} \\ &= \lim_{x \to \infty} \frac{(2/e^x)+3}{(7/e^x)-5}. \end{aligned}$$

As shown in **Figure 4.60**, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x \to \infty} \frac{2}{e^x} = 0 = \lim_{x \to \infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x \to \infty} f(x) = -\frac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \to \infty$. To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ to conclude that $\lim_{x \to \infty} f(x) = \frac{2}{7}$, and therefore the graph of approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \to -\infty$.

4.26 Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{(3e^x-4)}{(5e^x+2)}$.

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x - and y -intercepts.
3. Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \frac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as

$x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an *oblique asymptote*.

4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 4.28

Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1. Since f is a polynomial, the domain is the set of all real numbers.

Step 2. When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3. We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim_{x \to \infty} f(x) = \infty$. As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim_{x \to -\infty} f(x) = -\infty$. To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4. Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5. The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2$, $x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	**Test Point**	**Sign of Derivative** $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	**Conclusion**
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing.

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6. The second derivative of f is

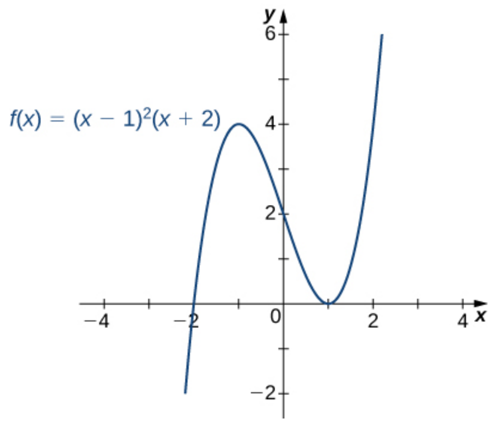
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	**Test Point**	**Sign of** $f''(x) = 6x$	**Conclusion**
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



4.27 Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Example 4.29

Sketching a Rational Function

Sketch the graph of $f(x) = \frac{x^2}{\left(1-x^2\right)}$.

Solution

Step 1. The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2. Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\frac{x^2}{\left(1-x^2\right)} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since f is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2. We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4. To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x \to 1^+} \frac{x^2}{1 - x^2} = -\infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{1 - x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x \to -1^+} \frac{x^2}{1 - x^2} = \infty \text{ and } \lim_{x \to -1^-} \frac{x^2}{1 - x^2} = -\infty.$$

Step 5. Calculate the first derivative:

$$f'(x) = \frac{\left(1 - x^2\right)(2x) - x^2(-2x)}{\left(1 - x^2\right)^2} = \frac{2x}{\left(1 - x^2\right)^2}.$$

Critical points occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	**Test Point**	**Sign of $f'(x) = \frac{2x}{\left(1 - x^2\right)^2}$**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

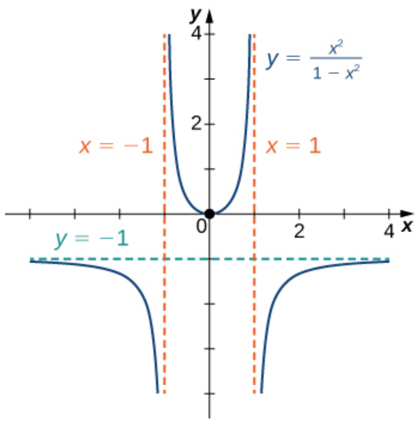
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{\left(1-x^2\right)^2(2) - 2x\left(2\left(1-x^2\right)(-2x)\right)}{\left(1-x^2\right)^4} \\ &= \frac{\left(1-x^2\right)\left[2\left(1-x^2\right)+8x^2\right]}{\left(1-x^2\right)^4} \\ &= \frac{2\left(1-x^2\right)+8x^2}{\left(1-x^2\right)^3} \\ &= \frac{6x^2+2}{\left(1-x^2\right)^3}. \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x)=0$ or $f''(x)$ is undefined. Since the numerator $6x^2+2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, -1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. in each of these intervals. The values $x=-2$, $x=0$, and $x=2$ are possible test points as shown in the following table.

Interval	**Test Point**	**Sign of $f''(x) = \frac{6x^2+2}{\left(1-x^2\right)^3}$**	**Conclusion**
$(-\infty, -1)$	$x=-2$	$+/- = -$	f is concave down.
$(-1, -1)$	$x=0$	$+/+ = +$	f is concave up.
$(1, \infty)$	$x=2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x=-1$ and $x=1$, there are no inflection points at either of these places because f is not continuous at $x=-1$ or $x=1$.



4.28 Sketch a graph of $f(x) = \frac{(3x + 5)}{(8 + 4x)}$.

Example 4.30

Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \frac{x^2}{(x - 1)}$

Solution

Step 1. The domain of f is the set of all real numbers x except $x = 1$.

Step 2. Find the intercepts. We can see that when $x = 0$, $f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x - 1} = x + 1 + \frac{1}{x - 1}.$$

Since $1/(x - 1) \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4. To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x \to 1^+} \frac{x^2}{x - 1} = \infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{x - 1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5. Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1$, $x = \frac{1}{2}$, $x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	**Test Point**	**Sign of** $f'(x) = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$	**Conclusion**
$(-\infty, 0)$	$x = -1$	$(-)(-)/+ = +$	f is increasing.
$(0, 1)$	$x = 1/2$	$(+)(-)/+ = -$	f is decreasing.
$(1, 2)$	$x = 3/2$	$(+)(-)/+ = -$	f is decreasing.
$(2, \infty)$	$x = 3$	$(+)(+)/+ = +$	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

Step 6. Calculate the second derivative:

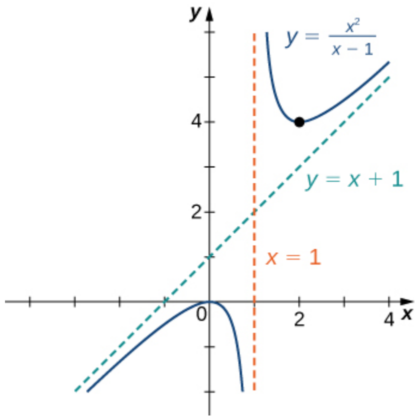
$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - \left(x^2 - 2x\right)(2(x-1))}{(x-1)^4} \\ &= \frac{(x-1)\left[(x-1)(2x-2) - 2\left(x^2 - 2x\right)\right]}{(x-1)^4} \\ &= \frac{(x-1)(2x-2) - 2\left(x^2 - 2x\right)}{(x-1)^3} \\ &= \frac{2x^2 - 4x + 2 - \left(2x^2 - 4x\right)}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$

and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{2}{(x-1)^3}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up.

From the information gathered, we arrive at the following graph for f.



4.29 Find the oblique asymptote for $f(x) = \frac{\left(3x^3 - 2x + 1\right)}{\left(2x^2 - 4\right)}$.

Example 4.31

Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$.

Solution

Step 1. Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = \left(\sqrt[3]{x-1}\right)^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim_{x \to \pm\infty} (x-1)^{2/3} = \infty,$ the function continues to grow without bound as $x \to \infty$ and $x \to -\infty.$

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate $f'.$ We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}.$$

This function is not zero anywhere, but it is undefined when $x = 1.$ Therefore, the only critical point is $x = 1.$ Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty),$ and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	**Test Point**	**Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$**	**Conclusion**
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

We conclude that f has a local minimum at $x = 1.$ Evaluating f at $x = 1,$ we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x \to 1} f'(x).$ Looking at the one-sided limits, we have

$$\lim_{x \to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1.$

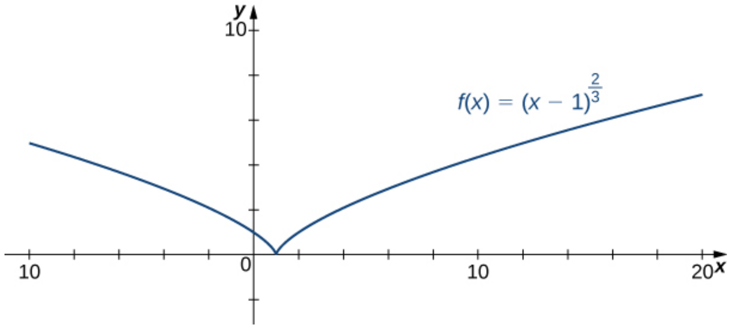
Step 6: To determine concavity, we calculate the second derivative of f:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all $x,$ but is undefined when $x = 1.$ Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty),$ and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	**Test Point**	**Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$**	**Conclusion**
$(-\infty, 1)$	$x = 0$	$-/+ = -$	f is concave down.
$(1, \infty)$	$x = 2$	$-/+ = -$	f is concave down.

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.

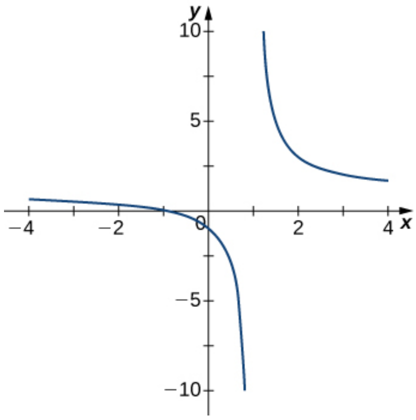


4.30 Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

4.6 EXERCISES

For the following exercises, examine the graphs. Identify where the vertical asymptotes are located.

251.



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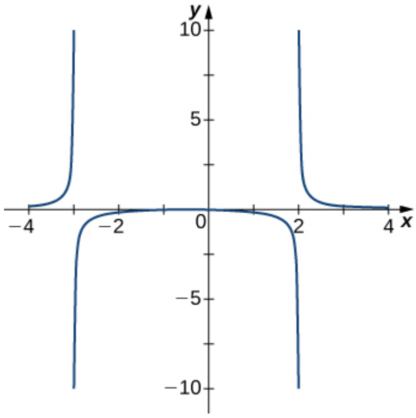
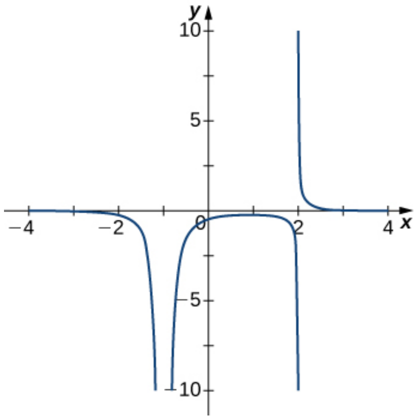


253.



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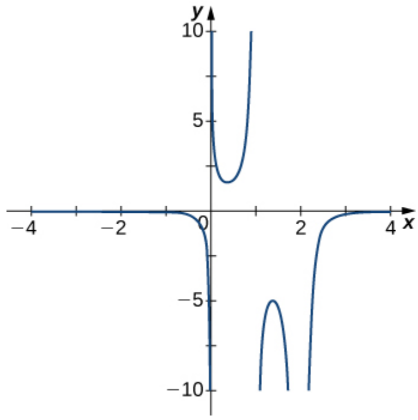
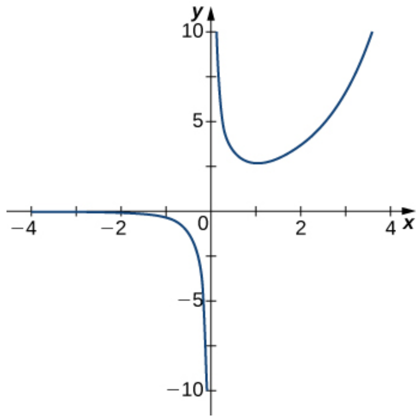


255.



For the following functions $f(x)$, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

256. $f(x) = \frac{x+1}{x^2+5x+4}$, $a = -1$

257. $f(x) = \frac{x}{x-2}$, $a = 2$

258. $f(x) = (x+2)^{3/2}$, $a = -2$

259. $f(x) = (x-1)^{-1/3}$, $a = 1$

260. $f(x) = 1 + x^{-2/5}$, $a = 1$

For the following exercises, evaluate the limit.

261. $\lim_{x \to \infty} \frac{1}{3x+6}$

262. $\lim_{x \to \infty} \frac{2x - 5}{4x}$

263. $\lim_{x \to \infty} \frac{x^2 - 2x + 5}{x + 2}$

264. $\lim_{x \to -\infty} \frac{3x^3 - 2x}{x^2 + 2x + 8}$

265. $\lim_{x \to -\infty} \frac{x^4 - 4x^3 + 1}{2 - 2x^2 - 7x^4}$

266. $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 1}}$

267. $\lim_{x \to -\infty} \frac{\sqrt{4x^2 - 1}}{x + 2}$

268. $\lim_{x \to \infty} \frac{4x}{\sqrt{x^2 - 1}}$

269. $\lim_{x \to -\infty} \frac{4x}{\sqrt{x^2 - 1}}$

270. $\lim_{x \to \infty} \frac{2\sqrt{x}}{x - \sqrt{x} + 1}$

For the following exercises, find the horizontal and vertical asymptotes.

271. $f(x) = x - \frac{9}{x}$

272. $f(x) = \frac{1}{1 - x^2}$

273. $f(x) = \frac{x^3}{4 - x^2}$

274. $f(x) = \frac{x^2 + 3}{x^2 + 1}$

275. $f(x) = \sin(x)\sin(2x)$

276. $f(x) = \cos x + \cos(3x) + \cos(5x)$

277. $f(x) = \frac{x\sin(x)}{x^2 - 1}$

278. $f(x) = \frac{x}{\sin(x)}$

279. $f(x) = \frac{1}{x^3 + x^2}$

280. $f(x) = \frac{1}{x - 1} - 2x$

281. $f(x) = \frac{x^3 + 1}{x^3 - 1}$

282. $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

283. $f(x) = x - \sin x$

284. $f(x) = \frac{1}{x} - \sqrt{x}$

For the following exercises, construct a function $f(x)$ that has the given asymptotes.

285. $x = 1$ and $y = 2$

286. $x = 1$ and $y = 0$

287. $y = 4, \quad x = -1$

288. $x = 0$

For the following exercises, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

289. **[T]** $f(x) = \frac{1}{x + 10}$

290. **[T]** $f(x) = \frac{x + 1}{x^2 + 7x + 6}$

291. **[T]** $\lim_{x \to -\infty} x^2 + 10x + 25$

292. **[T]** $\lim_{x \to -\infty} \frac{x + 2}{x^2 + 7x + 6}$

293. **[T]** $\lim_{x \to \infty} \frac{3x + 2}{x + 5}$

For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

294. $y = 3x^2 + 2x + 4$

295. $y = x^3 - 3x^2 + 4$

296. $y = \frac{2x + 1}{x^2 + 6x + 5}$

297. $y = \frac{x^3 + 4x^2 + 3x}{3x + 9}$

298. $y = \dfrac{x^2 + x - 2}{x^2 - 3x - 4}$

299. $y = \sqrt{x^2 - 5x + 4}$

300. $y = 2x\sqrt{16 - x^2}$

301. $y = \frac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

302. $y = e^x - x^3$

303. $y = x\tan x,\ x = [-\pi, \pi]$

304. $y = x\ln(x),\ x > 0$

305. $y = x^2\sin(x),\ x = [-2\pi, 2\pi]$

306. For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

307. For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$. must have what relation?

308. If $f'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

309. Both $f(x) = \dfrac{1}{(x-1)}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between these two functions?

310. True or false: Every ratio of polynomials has vertical asymptotes.

4.7 | Applied Optimization Problems

Learning Objectives

4.7.1 Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.32, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.32

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.62). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

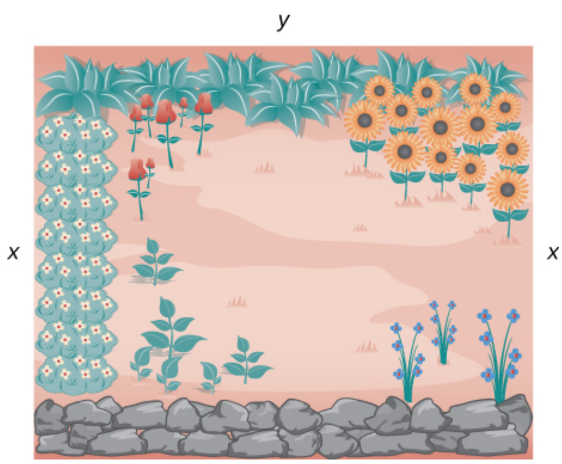


Figure 4.62 We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From **Figure 4.62**, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (**Figure 4.63**). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50\text{ ft}$. The area of this garden is 1250 ft^2.

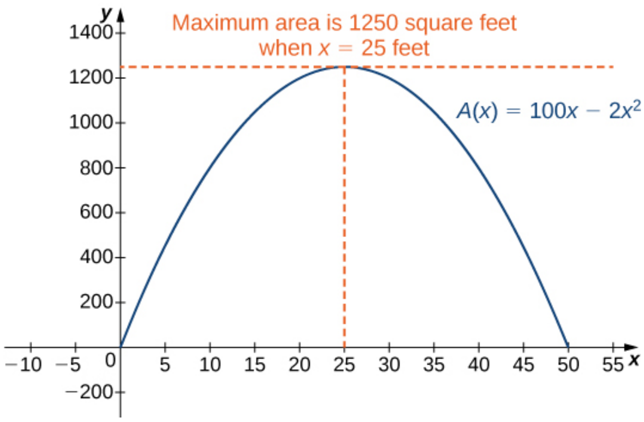


Figure 4.63 To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

4.31 Determine the maximum area if we want to make the same rectangular garden as in **Figure 4.63**, but we have 200 ft of fencing.

Now let's look at a general strategy for solving optimization problems similar to **Example 4.32**.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step $3.$ Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step $4.$ This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.33

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (**Figure 4.64**). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

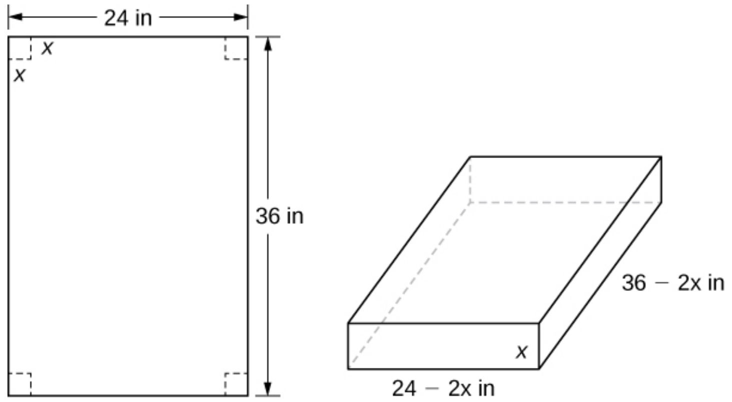


Figure 4.64 A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step $2,$ are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H,$ where $L,$ $W,$ and H are the length, width, and height, respectively.

Step 4: From **Figure 4.64**, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let's examine **Figure 4.64**. Certainly, we need $x > 0.$ Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12).$ Since V is a continuous function over the closed interval $[0, 12],$ we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12],$ V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12,$ the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by $12,$ the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825 \text{ in.}^3$ as shown in the following graph.

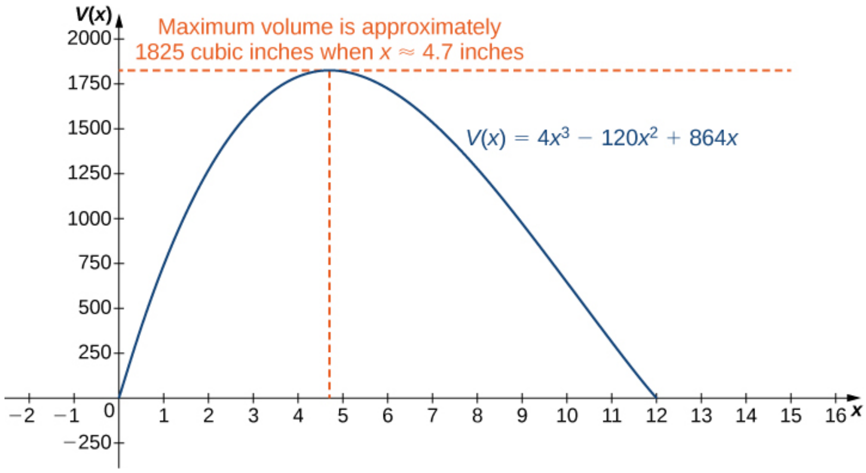


Figure 4.65 Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Watch a **video (http://www.openstax.org/l/20_boxvolume)** about optimizing the volume of a box.

4.32 Suppose the dimensions of the cardboard in **Example 4.33** are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Example 4.34

Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (**Figure 4.66**). Let T be the time it takes to get from the cabin to the island.

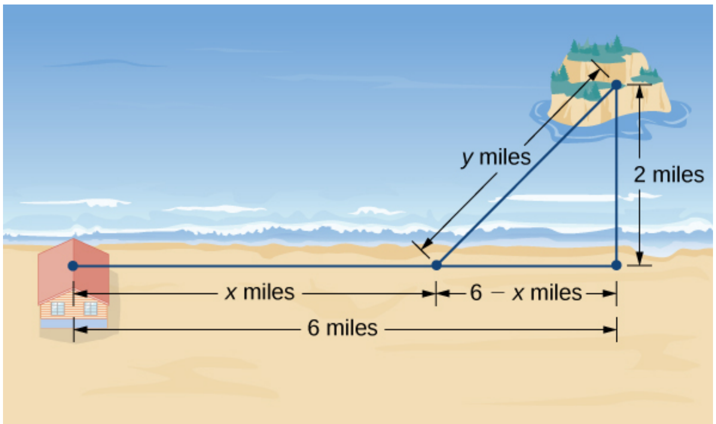


Figure 4.66 How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From **Figure 4.66**, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x\text{ mi}$. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Step 5: From **Figure 4.66**, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$T'(x) = \frac{1}{8} - \frac{1}{2}\frac{\left[(6 - x)^2 + 4\right]^{-1/2}}{3} \cdot 2(6 - x) = \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}}.$$

If $T'(x) = 0$, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}.$$

Therefore,

$$3\sqrt{(6-x)^2+4} = 8(6-x). \tag{4.6}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9\left[(6-x)^2+4\right] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of **Equation 4.6** to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies **Equation 4.6**. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas $T\left(6 - 6/\sqrt{55}\right) \approx 1.368$ h. Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

4.33 Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 4.35

Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function

$n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$R(p) = n \times p = (1000 - 5p)p = -5p^2 + 1000p.$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$ When $p = 100$, $R(100) = \$50{,}000$. When $p = 50$, $R(p) = \$37{,}500$. When $p = 200$, $R(p) = \$0$. Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

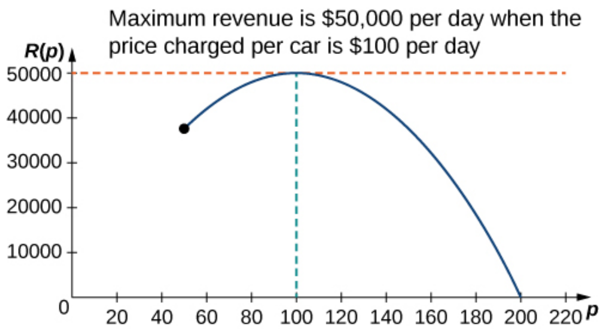


Figure 4.67 To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

4.34 A car rental company charges its customers p dollars per day, where $60 \le p \le 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Example 4.36

Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

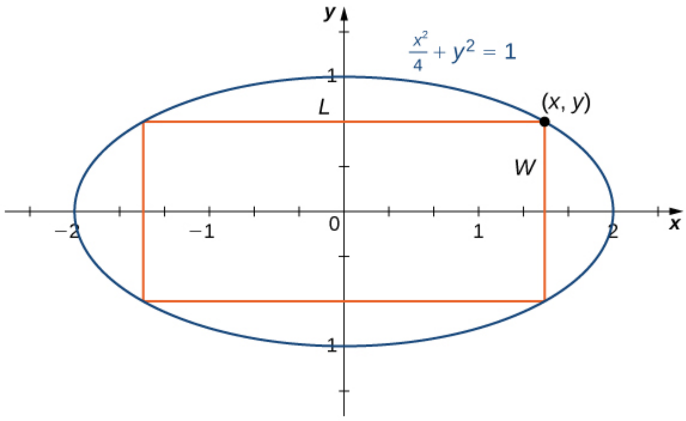


Figure 4.68 We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in **Figure 4.68**. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}.$$

Step 5: From **Figure 4.68**, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$. Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \\ &= \frac{8-4x^2}{\sqrt{4-x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8-4x^2}{\sqrt{4-x^2}} = 0, \tag{4.7}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of **Equation 4.7**. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of **Equation 4.7**. Since $x = \sqrt{2}$ is a solution of **Equation 4.7**, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$. Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

4.35 Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 4.37

Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\ \text{in.}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

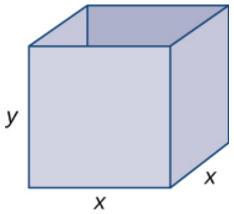


Figure 4.69 We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is $x^2 y$ and the volume is given as $216\ \text{in.}^3$, the constraint equation is

$$x^2 y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2 y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2 y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see **Figure 4.70**). When $x = 6\sqrt[3]{2}$, $y = \frac{216}{\left(6\sqrt[3]{2}\right)^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S\left(6\sqrt[3]{2}\right) = \frac{864}{6\sqrt[3]{2}} + \left(6\sqrt[3]{2}\right)^2 = 108\sqrt[3]{4}\text{ in.}^2$$

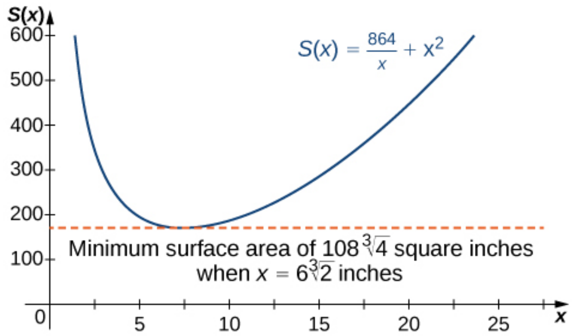


Figure 4.70 We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

4.36 Consider the same open-top box, which is to have volume 216 in.^3. Suppose the cost of the material for the base is $20\,¢/\text{in.}^2$ and the cost of the material for the sides is $30\,¢/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

4.7 EXERCISES

For the following exercises, answer by proof, counterexample, or explanation.

311. When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

312. Why do you need to check the endpoints for optimization problems?

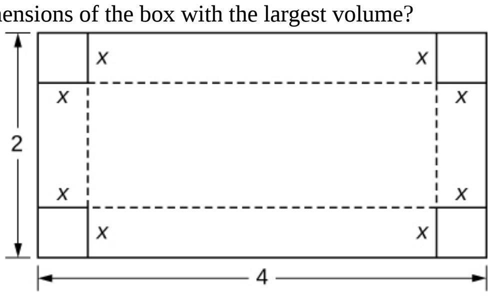
313. *True or False*. For every continuous nonlinear function, you can find the value x that maximizes the function.

314. *True or False*. For every continuous nonconstant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

315. To carry a suitcase on an airplane, the length $+\text{width}+$ height of the box must be less than or equal to $62\,\text{in}.$ Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right)h\right)^2.$ What height allows you to have the largest volume?

316. You are constructing a cardboard box with the dimensions $2\text{ m by }4\text{ m}.$ You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



317. Find the positive integer that minimizes the sum of the number and its reciprocal.

318. Find two positive integers such that their sum is $10,$ and minimize and maximize the sum of their squares.

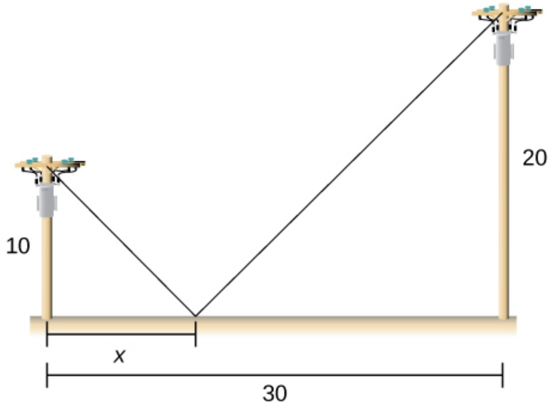
For the following exercises, consider the construction of a pen to enclose an area.

319. You have $400\,\text{ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

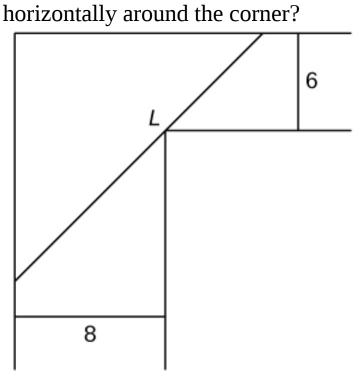
320. You have $800\,\text{ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

321. You need to construct a fence around an area of $1600\,\text{ft}.$ What are the dimensions of the rectangular pen to minimize the amount of material needed?

322. Two poles are connected by a wire that is also connected to the ground. The first pole is $20\,\text{ft}$ tall and the second pole is $10\,\text{ft}$ tall. There is a distance of $30\,\text{ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



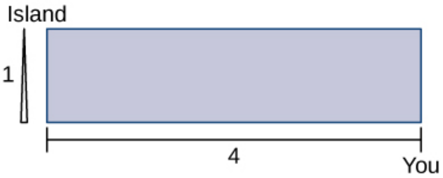
323. **[T]** You are moving into a new apartment and notice there is a corner where the hallway narrows from $8\text{ ft to }6\text{ ft}.$ What is the length of the longest item that can be carried horizontally around the corner?



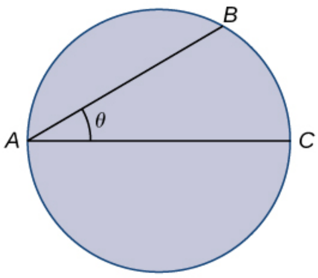
324. A patient's pulse measures $70\text{ bpm}, 80\text{ bpm}, \text{then }120\text{ bpm}.$ To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2 + (x-80)^2 + (x-120)^2?$ What value minimizes it?

325. In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

326. You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For the following problems, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w=3v$.



327. Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

328. Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

329. A truck uses gas as $g(v)=av+\frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

For the following exercises, consider a limousine that gets $m(v)=\frac{(120-2v)}{5}$ mi/gal at speed v, the chauffeur costs $\$15$/h, and gas is $\$3.5$/gal.

330. Find the cost per mile at speed v.

331. Find the cheapest driving speed.

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R(x)=ax$ and costs $C(x)=b+cx+dx^2$, where x represents the number of pizzas.

332. Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

333. Assume that $R(x)=10x$ and $C(x)=2x+x^2$. How many pizzas sold maximizes the profit?

334. Assume that $R(x)=15x$, and $C(x)=60+3x+\frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

335. Choose x to maximize the sum of their areas.

336. Choose x to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers x and y such that $x+y=10$. Maximize and minimize the quantities.

337. xy

338. x^2y^2

339. $y-\frac{1}{x}$

340. x^2-y

For the following exercises, draw the given optimization problem and solve.

341. Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

342. Find the volume of the largest right cone that fits in a sphere of radius 1.

343. Find the area of the largest rectangle that fits into the triangle with sides $x=0$, $y=0$ and $\frac{x}{4}+\frac{y}{6}=1$.

344. Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

345. Find the dimensions of the closed cylinder volume $V=16\pi$ that has the least amount of surface area.

346. Find the dimensions of a right cone with surface area $S=4\pi$ that has the largest volume.

For the following exercises, consider the points on the given graphs. Use a calculator to graph the functions.

347. **[T]** Where is the line $y = 5 - 2x$ closest to the origin?

348. **[T]** Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

349. **[T]** Where is the parabola $y = x^2$ closest to point $(2, 0)$?

350. **[T]** Where is the parabola $y = x^2$ closest to point $(0, 3)$?

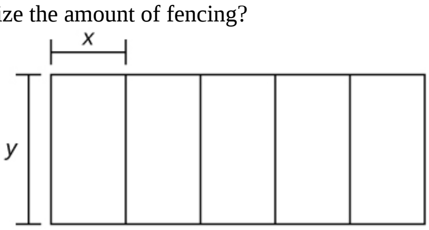
For the following exercises, set up, but do not evaluate, each optimization problem.

351. A window is composed of a semicircle placed on top of a rectangle. If you have $20\,\text{ft}$ of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.

352. You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

353. You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\,\text{ft}^2$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

354. You are building five identical pens adjacent to each other with a total area of $1000\,\text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



355. You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

4.8 | L’Hôpital’s Rule

Learning Objectives

4.8.1 Recognize when to apply L’Hôpital’s rule.

4.8.2 Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L’Hôpital’s rule in each case.

4.8.3 Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L’Hôpital’s rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L’Hôpital’s Rule

L’Hôpital’s rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2 \neq 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$? We call this one of the **indeterminate forms**, of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \text{ and } \lim_{x \to 0} \frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

For $\lim_{x \to 0} \frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more important, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L’Hôpital’s rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and such that $g'(a) \neq 0$ For x near a, we can write

$$f(x) \approx f(a) + f'(a)(x - a)$$

and

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$

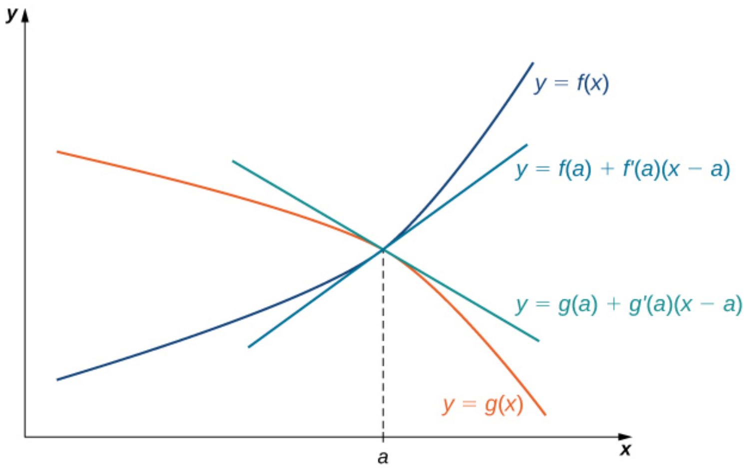


Figure 4.71 If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$ then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at $a,$ then f is continuous at $a,$ and therefore $f(a) = \lim_{x \to a} f(x) = 0.$ Similarly, $g(a) = \lim_{x \to a} g(x) = 0.$ If we also assume that f' and g' are continuous at $x = a,$ then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x).$ Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}.$ Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Theorem 4.12: L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing $a,$ except possibly at $a.$ If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty.$ This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty.$

Proof

We provide a proof of this theorem in the special case when $f, g, f',$ and g' are all continuous over an open interval containing $a.$ In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at $a,$ it follows that $f(a) = 0 = g(a).$ Therefore,

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{algebra} \\
&= \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} && \text{limit of a quotient} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\
&= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} && \text{continuity of } f' \text{ and } g' \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)}. && \text{limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.38

Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

b. $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x}$

c. $\lim_{x \to \infty} \frac{e^{1/x} - 1}{1/x}$

d. $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1 - \cos x \to 0$ and the denominator $x \to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(x)} \\
&= \lim_{x \to 0} \frac{\sin x}{1} \\
&= \frac{\lim_{x \to 0} (\sin x)}{\lim_{x \to 0} (1)} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

b. As $x \to 1,$ the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0.$ Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x} &= \lim_{x \to 1} \frac{\pi \cos(\pi x)}{1/x} \\ &= \lim_{x \to 1} (\pi x)\cos(\pi x) \\ &= (\pi \cdot 1)(-1) = -\pi.\end{aligned}$$

c. As $x \to \infty,$ the numerator $e^{1/x} - 1 \to 0$ and the denominator $\left(\frac{1}{x}\right) \to 0.$ Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to \infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0,$ both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0,$ we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0.$$

4.37 Evaluate $\lim_{x \to 0} \frac{x}{\tan x}.$

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty.$ Limits of this form are classified as *indeterminate forms of type* $\infty/\infty.$ Again, note that we are not actually dividing ∞ by $\infty.$ Since ∞ is not a real number, that is impossible; rather, $\infty/\infty.$ is used to represent a quotient of limits, each of which is ∞ or $-\infty.$

Theorem 4.13: L'Hôpital's Rule $(\infty/\infty$ Case)

Suppose f and g are differentiable functions over an open interval containing $a,$ except possibly at $a.$ Suppose $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty.$ This result also holds if the limit is infinite, if $a = \infty$ or

$-\infty$, or the limit is one-sided.

Example 4.39

Applying L'Hôpital's Rule $(\infty/\infty$ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to \infty} \frac{3x+5}{2x+1}$

b. $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x \to \infty}(3x+5) = \infty$ and $\lim_{x \to \infty}(2x+1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x \to \infty} \frac{3x+5}{2x+1/\ x} = \lim_{x \to \infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x \to \infty} \frac{3x+5}{2x+1} = \lim_{x \to \infty} \frac{3+5/x}{2x+1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x \to 0^+} \ln x = -\infty$ and $\lim_{x \to 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = \lim_{x \to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x \to 0^+} \frac{1}{-x\csc^2 x}.$$

Now as $x \to 0^+$, $\csc^2 x \to \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x \to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x \to 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x \to 0^+} \sin^2 x = 0$ and $\lim_{x \to 0^+} x = 0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x \to 0^+} \frac{\sin^2 x}{-x} = \lim_{x \to 0^+} \frac{2\sin x \cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = 0.$$

4.38 Evaluate $\lim_{x \to \infty} \frac{\ln x}{5x}$.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 4.40

When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x \to 1} \frac{x^2 + 5}{3x + 4}$. Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}\left(x^2 + 5\right) = 2x$$

and

$$\frac{d}{dx}(3x + 4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} = \lim_{x \to 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x \to 1}\left(x^2 + 5\right) = 6$ and $\lim_{x \to 1}(3x + 4) = 7,$ we actually have

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} = \frac{6}{7}.$$

We can conclude that

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} \neq \lim_{x \to 1} \frac{\frac{d}{dx}\left(x^2 + 5\right)}{\frac{d}{dx}(3x + 4)}.$$

4.39 Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$. Evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$ by other means.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a}(f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to a$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty$, $f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n + 1)}$ varies, depending on n. If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.41

Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln x = \frac{\ln x}{1/x},$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x \to 0^+} x \ln x = 0.$$

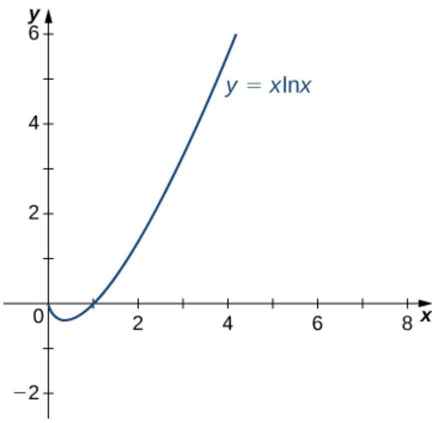


Figure 4.72 Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

4.40 Evaluate $\lim\limits_{x \to 0} x \cot x$.

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim\limits_{x \to \infty}(f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^3 - 3x^2 - 5\right) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^2 - 3x^2 - 5\right) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x - 3x^2 - 5\right) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.42

Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $\left(\sec^2 x\right) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \infty.$$

4.41 Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x)).$$

Therefore,

$$\lim_{x \to a}[\ln(y)] = \lim_{x \to a}[g(x)\ln(f(x))].$$

Since $\lim_{x \to a} f(x) = \infty$, we know that $\lim_{x \to a} \ln(f(x)) = \infty$. Therefore, $\lim_{x \to a} g(x)\ln(f(x))$ is of the indeterminate form

$0 \cdot \infty,$ and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x \to a} g(x)\ln(f(x)) = L,$ where L may be ∞ or $-\infty.$ Then

$$\lim_{x \to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to a} y\right) = L,$$

which gives us

$$\lim_{x \to a} y = \lim_{x \to a} f(x)^{g(x)} = e^L.$$

Example 4.43

Indeterminate Form of Type ∞^0

Evaluate $\lim_{x \to \infty} x^{1/x}.$

Solution

Let $y = x^{1/x}.$ Then,

$$\ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x \to \infty} \frac{\ln x}{x}.$ Applying L'Hôpital's rule, we obtain

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x \to \infty} \ln y = 0.$ Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to \infty} y\right) = 0,$$

which leads to

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,

$$\lim_{x \to \infty} x^{1/x} = 1.$$

4.42 Evaluate $\lim_{x \to \infty} x^{1/\ln(x)}.$

Example 4.44

Indeterminate Form of Type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x.$$

We now evaluate $\lim_{x \to 0^+} \sin x \ln x$. Since $\lim_{x \to 0^+} \sin x = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x \to 0^+} \frac{\cos x}{-1/\left(x(\ln x)^2\right)} = \lim_{x \to 0^+} \left(-x(\ln x)^2 \cos x\right).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x \to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x \to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim_{x \to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim_{x \to 0^+} y\right) = 0$ and we have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x \to 0^+} x^{\sin x} = 1.$$

4.43 Evaluate $\lim_{x \to 0^+} x^x$.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2.

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

Table 4.7 Comparing the Growth Rates of x^2 and x^3

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty. \text{ or, equivalently, } \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in the following table. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

Table 4.8 Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty; \text{ or, equivalently, } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.45

Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate $\lim\limits_{x \to \infty}\left(\dfrac{f(x)}{g(x)}\right)$.

a. $f(x) = x^2$ and $g(x) = e^x$

b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim\limits_{x \to \infty} x^2 = \infty$ and $\lim\limits_{x \to \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim\limits_{x \to \infty}\left[\dfrac{x^2}{e^x}\right]$. We obtain

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}.$$

Since $\lim\limits_{x \to \infty} 2x = \infty$ and $\lim\limits_{x \to \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See **Figure 4.73** and **Table 4.9**).

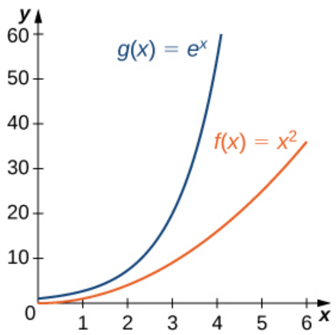


Figure 4.73 An exponential function grows at a faster rate than a power function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

Table 4.9
Growth rates of a power function and an exponential function.

b. Since $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see **Figure 4.74** and **Table 4.10**).

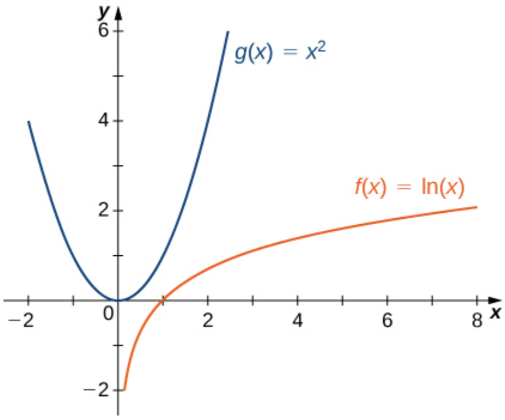


Figure 4.74 A power function grows at a faster rate than a logarithmic function.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Table 4.10
Growth rates of a power function and a logarithmic function

4.44 Compare the growth rates of x^{100} and 2^x.

Using the same ideas as in **Example 4.45**a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In **Figure 4.75** and **Table 4.11**, we compare e^x with x^3 and x^4 as $x \to \infty$.

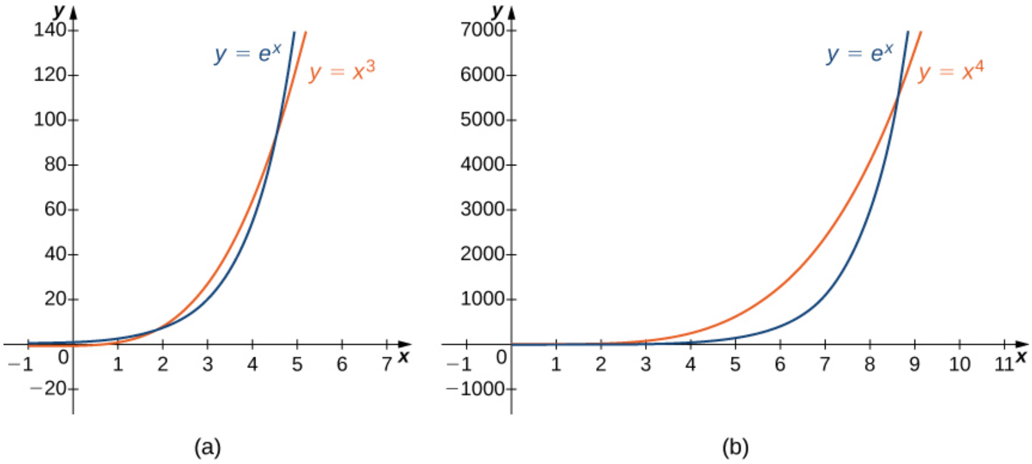


Figure 4.75 The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Table 4.11 An exponential function grows at a faster rate than any power function

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In **Figure 4.76** and **Table 4.12**, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

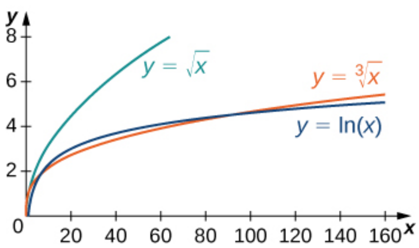


Figure 4.76 The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Table 4.12 A logarithmic function grows at a slower rate than any root function

4.8 EXERCISES

For the following exercises, evaluate the limit.

356. Evaluate the limit $\lim\limits_{x \to \infty} \frac{e^x}{x}$.

357. Evaluate the limit $\lim\limits_{x \to \infty} \frac{e^x}{x^k}$.

358. Evaluate the limit $\lim\limits_{x \to \infty} \frac{\ln x}{x^k}$.

359. Evaluate the limit $\lim\limits_{x \to a} \frac{x-a}{x^2-a^2}, \quad a \neq 0$.

360. Evaluate the limit $\lim\limits_{x \to a} \frac{x-a}{x^3-a^3}, \quad a \neq 0$.

361. Evaluate the limit $\lim\limits_{x \to a} \frac{x-a}{x^n-a^n}, \quad a \neq 0$.

For the following exercises, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

362. $\lim\limits_{x \to 0^+} x^2 \ln x$

363. $\lim\limits_{x \to \infty} x^{1/x}$

364. $\lim\limits_{x \to 0} x^{2/x}$

365. $\lim\limits_{x \to 0} \frac{x^2}{1/x}$

366. $\lim\limits_{x \to \infty} \frac{e^x}{x}$

For the following exercises, evaluate the limits with either L'Hôpital's rule or previously learned methods.

367. $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$

368. $\lim\limits_{x \to 3} \frac{x^2-9}{x+3}$

369. $\lim\limits_{x \to 0} \frac{(1+x)^{-2}-1}{x}$

370. $\lim\limits_{x \to \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$

371. $\lim\limits_{x \to \pi} \frac{x-\pi}{\sin x}$

372. $\lim\limits_{x \to 1} \frac{x-1}{\sin x}$

373. $\lim\limits_{x \to 0} \frac{(1+x)^n-1}{x}$

374. $\lim\limits_{x \to 0} \frac{(1+x)^n-1-nx}{x^2}$

375. $\lim\limits_{x \to 0} \frac{\sin x - \tan x}{x^3}$

376. $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

377. $\lim\limits_{x \to 0} \frac{e^x-x-1}{x^2}$

378. $\lim\limits_{x \to 0} \frac{\tan x}{\sqrt{x}}$

379. $\lim\limits_{x \to 1} \frac{x-1}{\ln x}$

380. $\lim\limits_{x \to 0} (x+1)^{1/x}$

381. $\lim\limits_{x \to 1} \frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

382. $\lim\limits_{x \to 0^+} x^{2x}$

383. $\lim\limits_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

384. $\lim\limits_{x \to 0} \frac{\sin x - x}{x^2}$

385. $\lim\limits_{x \to 0^+} x \ln\left(x^4\right)$

386. $\lim\limits_{x \to \infty} (x-e^x)$

387. $\lim\limits_{x \to \infty} x^2 e^{-x}$

388. $\lim\limits_{x \to 0} \frac{3^x-2^x}{x}$

389. $\lim\limits_{x \to 0} \frac{1+1/x}{1-1/x}$

390. $\lim\limits_{x \to \pi/4} (1-\tan x)\cot x$

391. $\lim_{x \to \infty} xe^{1/x}$

392. $\lim_{x \to 0^+} x^{1/\cos x}$

393. $\lim_{x \to 0^+} x^{1/x}$

394. $\lim_{x \to 0^-} \left(1 - \frac{1}{x}\right)^x$

395. $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

For the following exercises, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

396. **[T]** $\lim_{x \to 0} \frac{e^x - 1}{x}$

397. **[T]** $\lim_{x \to 0} x\sin\left(\frac{1}{x}\right)$

398. **[T]** $\lim_{x \to 1} \frac{x - 1}{1 - \cos(\pi x)}$

399. **[T]** $\lim_{x \to 1} \frac{e^{(x - 1)} - 1}{x - 1}$

400. **[T]** $\lim_{x \to 1} \frac{(x - 1)^2}{\ln x}$

401. **[T]** $\lim_{x \to \pi} \frac{1 + \cos x}{\sin x}$

402. **[T]** $\lim_{x \to 0} \left(\csc x - \frac{1}{x}\right)$

403. **[T]** $\lim_{x \to 0^+} \tan(x^x)$

404. **[T]** $\lim_{x \to 0^+} \frac{\ln x}{\sin x}$

405. **[T]** $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

4.9 | Newton's Method

Learning Objectives

4.9.1 Describe the steps of Newton's method.
4.9.2 Explain what an iterative process means.
4.9.3 Recognize when Newton's method does not work.
4.9.4 Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0.$ For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the zeroes of functions. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0.$ If f is the first-degree polynomial $f(x) = ax + b,$ then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}.$ If f is the second-degree polynomial $f(x) = ax^2 + bx + c,$ the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0.$ Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0.$ No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0.$ By sketching a graph of $f,$ we can estimate a root of $f(x) = 0.$ Let's call this estimate $x_0.$ We then draw the tangent line to f at $x_0.$ If $f'(x_0) \neq 0,$ this tangent line intersects the x-axis at some point $(x_1, 0).$ Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at $x_1.$ If $f'(x_1) \neq 0,$ this tangent line also intersects the x-axis, producing another approximation, $x_2.$ We continue in this way, deriving a list of approximations: $x_0, x_1, x_2, \ldots.$ Typically, the numbers $x_0, x_1, x_2, \ldots$ quickly approach an actual root $x^*,$ as shown in the following figure.

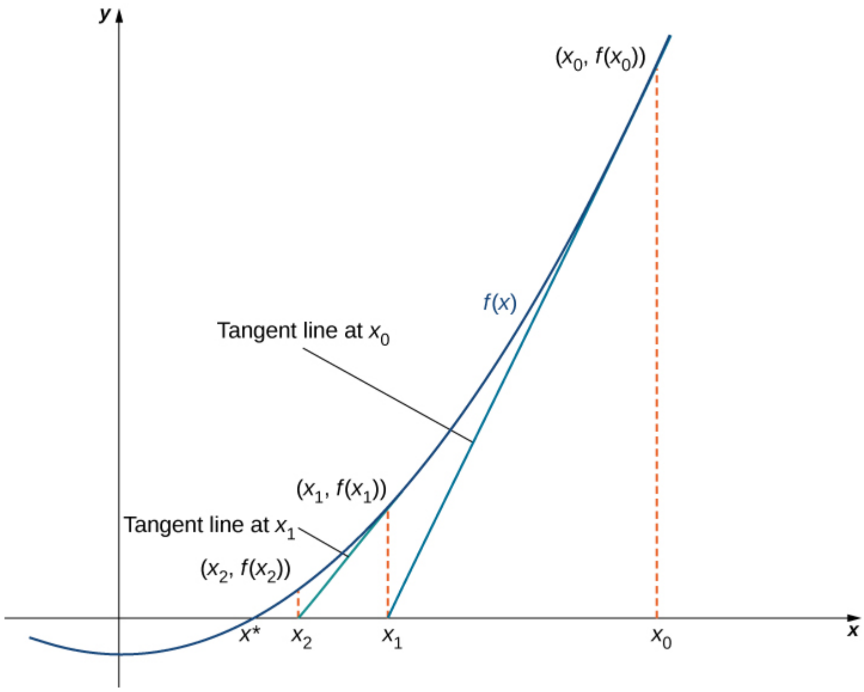


Figure 4.77 The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{4.8}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 4.46

Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1, x_2, x_3, x_4,$ and x_5.

Solution

From **Figure 4.78**, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use **Equation 4.8**. Since $f(x) = x^3 - 3x + 1,$ the derivative is $f'(x) = 3x^2 - 3$. Using **Equation 4.8** with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, $x_2,$ we use **Equation 4.8** with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 = \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

$$\begin{aligned} x_1 &\approx 1.666666667 \\ x_2 &\approx 1.548611111 \\ x_3 &\approx 1.532390162 \\ x_4 &\approx 1.532088989 \\ x_5 &\approx 1.532088886 \\ x_6 &\approx 1.532088886. \end{aligned}$$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

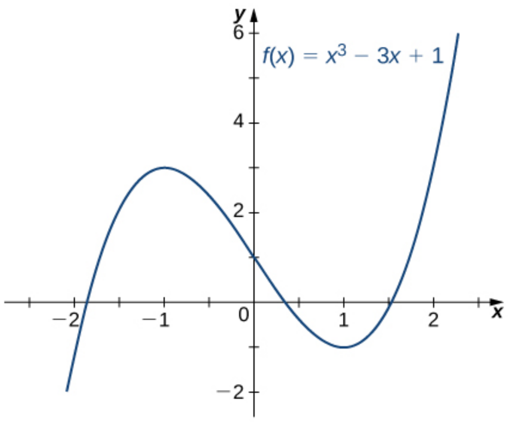


Figure 4.78 The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

4.45 Letting $x_0 = 0,$ let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0,\ 1]$ by calculating x_1 and $x_2.$

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}.$ This method can be modified to approximate the square root of any positive number.

Example 4.47

Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (**Figure 4.79**). Let $f(x) = x^2 - 2,$ let $x_0 = 2,$ and calculate $x_1, x_2, x_3, x_4, x_5.$ (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2},$ the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}.$)

Solution

For $f(x) = x^2 - 2,\ f'(x) = 2x.$ From **Equation 4.8**, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x^2{}_{n-1} - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5 \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667. \end{aligned}$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562. \end{aligned}$$

Since we obtained the same value for x_4 and $x_5,$ it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562.$

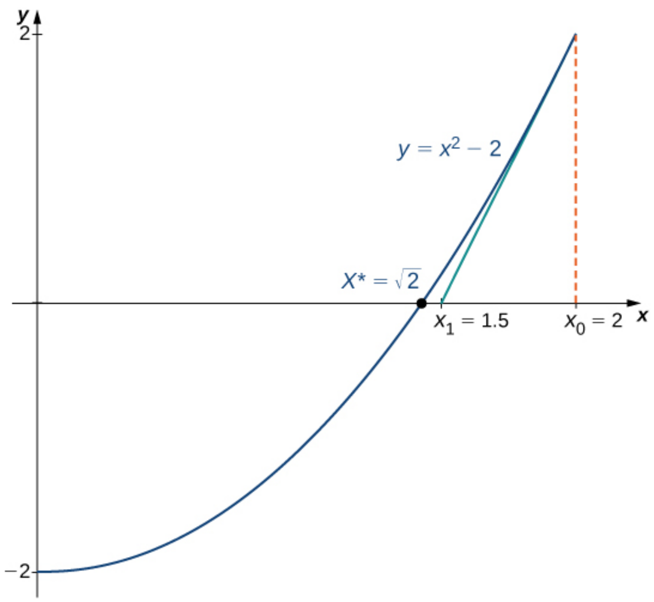


Figure 4.79 We can use Newton's method to find $\sqrt{2}$.

4.46 Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite **Equation 4.8** as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see **Figure 4.80**). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In **Example 4.48**, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

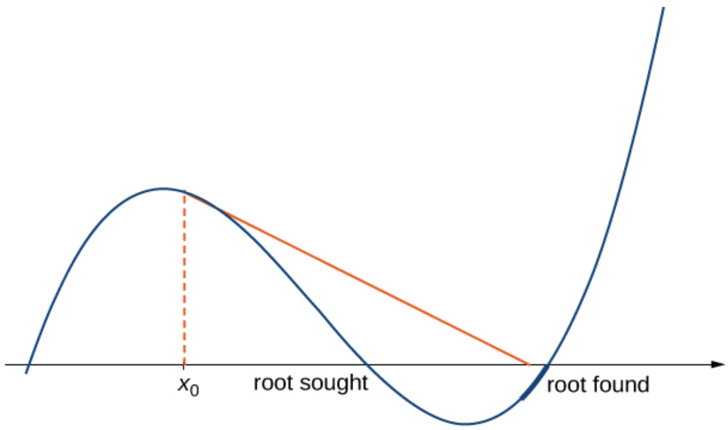


Figure 4.80 If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.48

When Newton’s Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (see **Figure 4.81**). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

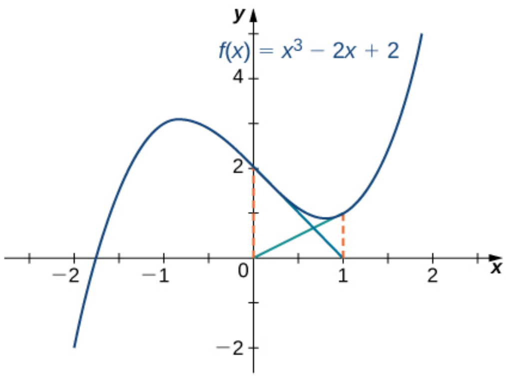


Figure 4.81 The approximations continue to alternate between 0 and 1 and never approach the root of f.

4.47 For $f(x) = x^3 - 2x + 2,$ let $x_0 = -1.5$ and find x_1 and x_2.

From **Example 4.48**, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In **Example 4.49**, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 4.49

Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1,$ let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3 \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0,$ then

$$\begin{aligned}
x_1 &= \frac{1}{2}(0) + 4 = 4 \\
x_2 &= \frac{1}{2}(4) + 4 = 6 \\
x_3 &= \frac{1}{2}(6) + 4 = 7 \\
x_4 &= \frac{1}{2}(7) + 4 = 7.5 \\
x_5 &= \frac{1}{2}(7.5) + 4 = 7.75 \\
x_6 &= \frac{1}{2}(7.75) + 4 = 7.875 \\
x_7 &= \frac{1}{2}(7.875) + 4 = 7.9375 \\
x_8 &= \frac{1}{2}(7.9375) + 4 = 7.96875 \\
x_9 &= \frac{1}{2}(7.96875) + 4 = 7.984375.
\end{aligned}$$

From this list, we conjecture that the values x_n approach 8.

Figure 4.82 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty$.

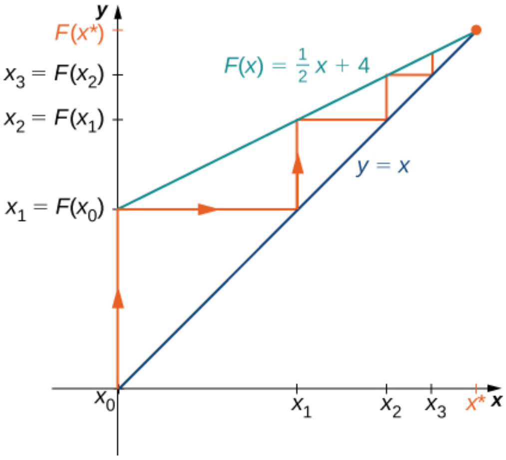


Figure 4.82 This iterative process approaches the value $x^* = 8$.

4.48 Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Student PROJECT

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in **Example 4.48** that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-*cycle*. Iterative processes can converge to cycles with various periodicities, such as $2-\text{cycles},\ 4-\text{cycles}$ (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call *chaos*. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the Mandelbrot set (see **Figure 4.83**), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

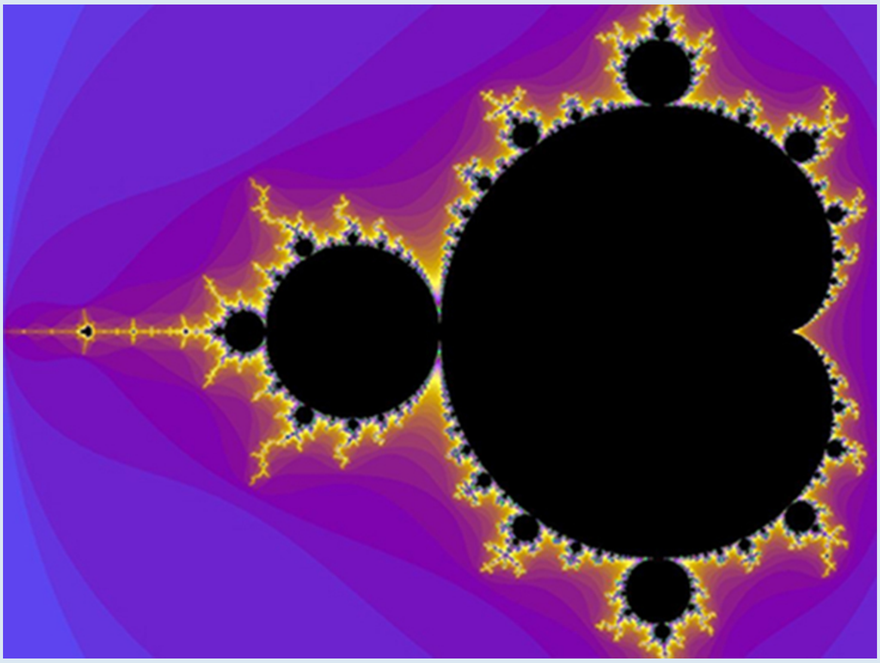


Figure 4.83 The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1-x), \text{ where } x \in [0,\ 1] \text{ and } r > 0$$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of $r,$ the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a *cobweb diagram*. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. **Figure 4.84** shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

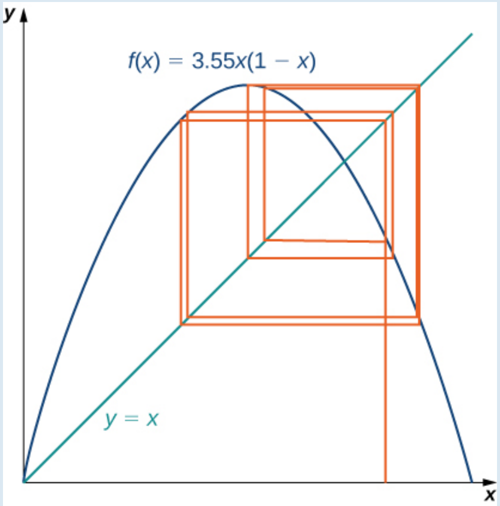


Figure 4.84 A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, $2 - \text{cycle}$, $4 - \text{cycle}.$)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

4.9 EXERCISES

For the following exercises, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

406. $f(x) = x^2 + 1$

407. $f(x) = x^3 + 2x + 1$

408. $f(x) = \sin x$

409. $f(x) = e^x$

410. $f(x) = x^3 + 3xe^x$

For the following exercises, solve $f(x) = 0$ using the iteration $x_{n+1} = x_n - cf(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

411. $f(x) = x^2 - 4$, with $x_0 = 0$

412. $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

413. What is the value of "c" for Newton's method?

For the following exercises, start at

a. $x_0 = 0.6$ and

b. $x_0 = 2$.

Compute x_1 and x_2 using the specified iterative method.

414. $x_{n+1} = x_n{}^2 - \frac{1}{2}$

415. $x_{n+1} = 2x_n(1 - x_n)$

416. $x_{n+1} = \sqrt{x_n}$

417. $x_{n+1} = \frac{1}{\sqrt{x_n}}$

418. $x_{n+1} = 3x_n(1 - x_n)$

419. $x_{n+1} = x_n{}^2 + x_n - 2$

420. $x_{n+1} = \frac{1}{2}x_n - 1$

421. $x_{n+1} = |x_n|$

For the following exercises, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

422. $x^2 - 10 = 0$

423. $x^4 - 100 = 0$

424. $x^2 - x = 0$

425. $x^3 - x = 0$

426. $x + 5\cos(x) = 0$

427. $x + \tan(x) = 0$, choose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

428. $\frac{1}{1 - x} = 2$

429. $1 + x + x^2 + x^3 + x^4 = 2$

430. $x^3 + (x + 1)^3 = 10^3$

431. $x = \sin^2(x)$

For the following exercises, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

432. $\sin x$

433. $\tan(x)$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

434. $e^x - 2$

435. $\ln(x) + 2$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. For the following exercises, consider the formulation of the method.

436. To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$.

437. What additional restrictions are necessary on the function f?

For the following exercises, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

438. Minimum of $f(x) = x^2 + 2x + 4$

439. Minimum of $f(x) = 3x^3 + 2x^2 - 16$

440. Minimum of $f(x) = x^2 e^x$

441. Maximum of $f(x) = x + \frac{1}{x}$

442. Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

443. Maximum of $f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$

444. Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

445. Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

For the following exercises, use the specified method to solve the equation. If it does not work, explain why it does not work.

446. Newton's method, $x^2 + 2 = 0$

447. Newton's method, $0 = e^x$

448. Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$

449. Solving $x_{n+1} = -x_n{}^3$ starting at $x_0 = -1$

For the following exercises, use the secant method, an alternative iterative method to Newton's method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

450. Find a root to $0 = x^2 - x - 3$ accurate to three decimal places.

451. Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

452. Find a root to $0 = e^x - 2$ accurate to four decimal places.

453. Find a root to $\ln(x + 2) = \frac{1}{2}$ accurate to four decimal places.

454. Why would you use the secant method over Newton's method? What are the necessary restrictions on f?

For the following exercises, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

455. $f(x) = x^2 + 2x + 1,\ x_0 = 1$

456. $f(x) = x^2,\ x_0 = 1$

457. $f(x) = \sin x,\ x_0 = 1$

458. $f(x) = e^x - 1,\ x_0 = 2$

459. $f(x) = x^3 + 2x + 4,\ x_0 = 0$

In the following exercises, consider Kepler's equation regarding planetary orbits, $M = E - \varepsilon \sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

460. Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon = 0.25$; round to three decimals.

461. Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon = 0.8$; round to three decimals.

The following two exercises consider a bank investment. The initial investment is $\$10{,}000$. After 25 years, the investment has tripled to $\$30{,}000$.

462. Use Newton's method to determine the interest rate if the interest was compounded annually.

463. Use Newton's method to determine the interest rate if the interest was compounded continuously.

464. The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \left(\frac{1}{2}\right)x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for $\$20$.

4.10 | Antiderivatives

Learning Objectives

4.10.1 Find the general antiderivative of a given function.
4.10.2 Explain the terms and notation used for an indefinite integral.
4.10.3 State the power rule for integrals.
4.10.4 Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in **Derivatives** of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (**Introduction to Techniques of Integration (http://cnx.org/content/m53654/latest/)**).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f, we call F an antiderivative of f.

Definition

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 4.14: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

i. for each constant C, the function $F(x)+C$ is also an antiderivative of f over I;

ii. if G is an antiderivative of f over I, there is a constant C for which $G(x)=F(x)+C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x)+C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.50

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x)=3x^2$

b. $f(x)=\frac{1}{x}$

c. $f(x)=\cos x$

d. $f(x)=e^x$

Solution

a. Because

$$\frac{d}{dx}\left(x^3\right)=3x^2$$

then $F(x)=x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form x^3+C for some constant C, and every function of the form x^3+C is an antiderivative of $3x^2$.

b. Let $f(x)=\ln|x|$. For $x>0$, $f(x)=\ln(x)$ and

$$\frac{d}{dx}(\ln x)=\frac{1}{x}.$$

For $x<0$, $f(x)=\ln(-x)$ and

$$\frac{d}{dx}(\ln(-x))=-\frac{1}{-x}=\frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln|x|)=\frac{1}{x}.$$

Thus, $F(x)=\ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x|+C$ for some constant C and every function of the form $\ln|x|+C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

4.49 Find all antiderivatives of $f(x) = \sin x$.

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is called an *integral sign*, and $\int f(x)dx$ is called the indefinite integral of f.

Definition

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as *integrating* f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as *the family of antiderivatives of* f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the *family of antiderivatives of* $2x$. **Figure 4.85** shows a graph of this family of antiderivatives.

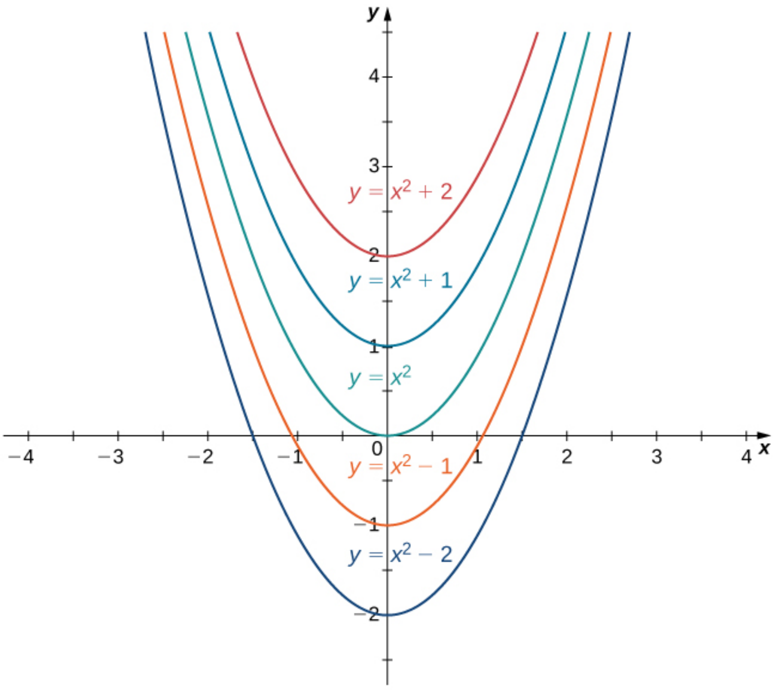


Figure 4.85 The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as *the power rule for integrals*.

Theorem 4.15: Power Rule for Integrals

For $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in **Appendix B**.

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int kx^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dn = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1}\lvert x\rvert\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}\lvert x\rvert + C$

Table 4.13 Integration Formulas

From the definition of indefinite integral of f, we know

$$\int f(x)dx = F(x) + C$$

if and only if F is an antiderivative of f. Therefore, when claiming that

$$\int f(x)dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.51

Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x\,dx = xe^x - e^x + C$

Solution

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2 e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2 e^x}{2}\right) = xe^x + \frac{x^2 e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

4.50 Verify that $\int x\cos x\,dx = x\sin x + \cos x + C.$

In **Table 4.13**, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g.$ In **Example 4.51**a. we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\left(\frac{x^2}{2}\right) + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and $g,$ respectively, then

$$\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int (f(x) + g(x))dx = F(x) + G(x) + C.$$

Similarly,

$$\int (f(x) - g(x))dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x),$ where k is any real number. Since

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}F(x) = kf'(x)$$

for any real number $k,$ we conclude that

$$\int kf(x)dx = kF(x) + C.$$

These properties are summarized next.

Theorem 4.16: Properties of Indefinite Integrals

Let F and G be antiderivatives of f and $g,$ respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see **Example 4.51**b. for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in **Introduction to Integration**. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.52

Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int (5x^3 - 7x^2 + 3x + 4)dx$

b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x} dx$

c. $\int \frac{4}{1 + x^2} dx$

d. $\int \tan x \cos x \, dx$

Solution

a. Using **Properties of Indefinite Integrals**, we can integrate each of the four terms in the integrand separately. We obtain

$$\int (5x^3 - 7x^2 + 3x + 4)dx = \int 5x^3 \, dx - \int 7x^2 \, dx + \int 3x \, dx + \int 4 dx.$$

From the second part of **Properties of Indefinite Integrals**, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3 \, dx - \int 7x^2 \, dx + \int 3x \, dx + \int 4 dx = 5\int x^3 \, dx - 7\int x^2 \, dx + 3\int x \, dx + 4\int 1 dx.$$

Using the power rule for integrals, we conclude that

$$\int (5x^3 - 7x^2 + 3x + 4)dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right)dx &= \int x \, dx + 4\int x^{-2/3} dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right) + 1}x^{(-2/3) + 1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using **Properties of Indefinite Integrals**, write the integral as

$$4\int \frac{1}{1 + x^2} dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{(1 + x^2)}$ to conclude that

$$\int \frac{4}{1 + x^2} dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x}\cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

4.51 Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) dx.$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A *differential equation* is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.9}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of **Equation 4.9** are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2 \, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0)—that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \tag{4.10}$$

with the additional condition

$$y(x_0) = y_0 \tag{4.11}$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C,$ to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5.$ From this equation, we see that $C = 3,$ and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

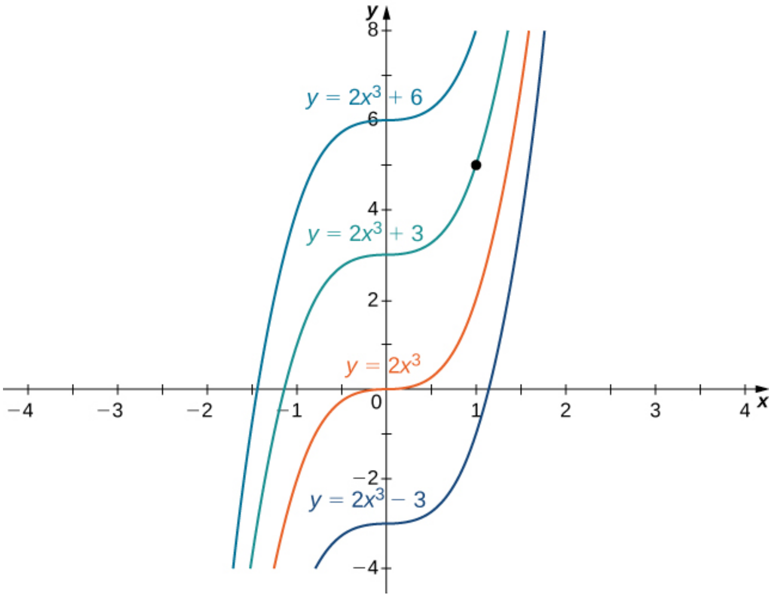


Figure 4.86 Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 4.53

Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \; y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x,$ then

$$y = \int \sin(x) dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5.$ Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6.$

4.52 Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, \; y(1) = 2.$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car.

We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.54

Decelerating Car

A car is traveling at the rate of 88 ft/sec $(60$ mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.
The car is traveling at a rate of $88\,\text{ft/sec}$. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is
$$a(t) = -15\text{ ft/s}^2.$$
The acceleration is the derivative of the velocity,
$$v'(t) = -15.$$
Therefore, we have an initial-value problem to solve:
$$v'(t) = -15,\ v(0) = 88.$$
Integrating, we find that
$$v(t) = -15t + C.$$
Since $v(0) = 88,\ C = 88.$ Thus, the velocity function is
$$v(t) = -15t + 88.$$
To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0,$ we obtain $t = \frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem
$$s'(t) = -15t + 88,\ s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0,$ the constant is $C = 0.$ Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

4.53 Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

4.10 EXERCISES

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

465. $F(x) = 5x^3 + 2x^2 + 3x + 1,\ f(x) = 15x^2 + 4x + 3$

466. $F(x) = x^2 + 4x + 1,\ f(x) = 2x + 4$

467. $F(x) = x^2 e^x,\ f(x) = e^x\left(x^2 + 2x\right)$

468. $F(x) = \cos x,\ f(x) = -\sin x$

469. $F(x) = e^x,\ f(x) = e^x$

For the following exercises, find the antiderivative of the function.

470. $f(x) = \frac{1}{x^2} + x$

471. $f(x) = e^x - 3x^2 + \sin x$

472. $f(x) = e^x + 3x - x^2$

473. $f(x) = x - 1 + 4\sin(2x)$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

474. $f(x) = 5x^4 + 4x^5$

475. $f(x) = x + 12x^2$

476. $f(x) = \frac{1}{\sqrt{x}}$

477. $f(x) = (\sqrt{x})^3$

478. $f(x) = x^{1/3} + (2x)^{1/3}$

479. $f(x) = \frac{x^{1/3}}{x^{2/3}}$

480. $f(x) = 2\sin(x) + \sin(2x)$

481. $f(x) = \sec^2(x) + 1$

482. $f(x) = \sin x \cos x$

483. $f(x) = \sin^2(x)\cos(x)$

484. $f(x) = 0$

485. $f(x) = \frac{1}{2}\csc^2(x) + \frac{1}{x^2}$

486. $f(x) = \csc x \cot x + 3x$

487. $f(x) = 4\csc x \cot x - \sec x \tan x$

488. $f(x) = 8\sec x(\sec x - 4\tan x)$

489. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

For the following exercises, evaluate the integral.

490. $\int(-1)dx$

491. $\int \sin x\,dx$

492. $\int(4x + \sqrt{x})dx$

493. $\int \frac{3x^2 + 2}{x^2}dx$

494. $\int(\sec x \tan x + 4x)dx$

495. $\int\left(4\sqrt{x} + \sqrt[4]{x}\right)dx$

496. $\int\left(x^{-1/3} - x^{2/3}\right)dx$

497. $\int \frac{14x^3 + 2x + 1}{x^3}dx$

498. $\int(e^x + e^{-x})dx$

For the following exercises, solve the initial value problem.

499. $f'(x) = x^{-3},\ f(1) = 1$

500. $f'(x) = \sqrt{x} + x^2,\ f(0) = 2$

501. $f'(x) = \cos x + \sec^2(x),\ f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$

502. $f'(x) = x^3 - 8x^2 + 16x + 1,\ f(0) = 0$

503. $f'(x) = \frac{2}{x^2} - \frac{x^2}{2},\ f(1) = 0$

For the following exercises, find two possible functions f given the second- or third-order derivatives.

504. $f''(x) = x^2 + 2$

505. $f''(x) = e^{-x}$

506. $f''(x) = 1 + x$

507. $f'''(x) = \cos x$

508. $f'''(x) = 8e^{-2x} - \sin x$

509. A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec^2. How long before the car stops?

510. In the preceding problem, calculate how far the car travels in the time it takes to stop.

511. You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec^2. How long does it take you to reach merging speed at 60 mph?

512. Based on the previous problem, how far does the car travel to reach merging speed?

513. A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

514. A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

For the following exercises, find the antiderivative of the function, assuming $F(0) = 0$.

515. **[T]** $f(x) = x^2 + 2$

516. **[T]** $f(x) = 4x - \sqrt{x}$

517. **[T]** $f(x) = \sin x + 2x$

518. **[T]** $f(x) = e^x$

519. **[T]** $f(x) = \frac{1}{(x+1)^2}$

520. **[T]** $f(x) = e^{-2x} + 3x^2$

For the following exercises, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

521. If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

522. If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

523. If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

524. If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

CHAPTER 4 REVIEW

KEY TERMS

absolute extremum if f has an absolute maximum or absolute minimum at $c,$ we say f has an absolute extremum at c

absolute maximum if $f(c) \geq f(x)$ for all x in the domain of $f,$ we say f has an absolute maximum at c

absolute minimum if $f(c) \leq f(x)$ for all x in the domain of $f,$ we say f has an absolute minimum at c

antiderivative a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

concave down if f is differentiable over an interval I and f' is decreasing over $I,$ then f is concave down over I

concave up if f is differentiable over an interval I and f' is increasing over $I,$ then f is concave up over I

concavity the upward or downward curve of the graph of a function

concavity test suppose f is twice differentiable over an interval I; if $f'' > 0$ over $I,$ then f is concave up over I; if $f'' < 0$ over $I,$ then f is concave down over I

critical point if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

differential the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)dx$

differential form given a differentiable function $y = f'(x),$ the equation $dy = f'(x)dx$ is the differential form of the derivative of y with respect to x

end behavior the behavior of a function as $x \to \infty$ and $x \to -\infty$

extreme value theorem if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem if f has a local extremum at $c,$ then c is a critical point of f

first derivative test let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through $c,$ then f has a local maximum at c; if f' changes sign from negative to positive as x increases through $c,$ then f has a local minimum at c; if f' does not change sign as x increases through $c,$ then f does not have a local extremum at c

horizontal asymptote if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L,$ then $y = L$ is a horizontal asymptote of f

indefinite integral the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)dx$ to denote the indefinite integral of f

indeterminate forms when evaluating a limit, the forms $\frac{0}{0},$ $\infty/\infty,$ $0 \cdot \infty,$ $\infty - \infty,$ $0^0,$ $\infty^0,$ and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is

infinite limit at infinity a function that becomes arbitrarily large as x becomes large

inflection point if f is continuous at c and f changes concavity at $c,$ the point $(c, f(c))$ is an inflection point of f

initial value problem a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

iterative process process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

limit at infinity the limiting value, if it exists, of a function as $x \to \infty$ or $x \to -\infty$

linear approximation the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

local extremum if f has a local maximum or local minimum at $c,$ we say f has a local extremum at c

local maximum if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I,$ we say f has a local maximum at c

local minimum if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I,$ we say f has a local minimum at c

L'Hôpital's rule if f and g are differentiable functions over an interval $a,$ except possibly at $a,$ and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$ assuming the limit on the right exists or is ∞ or $-\infty$

mean value theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Newton's method method for approximating roots of $f(x) = 0;$ using an initial guess $x_0;$ each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

oblique asymptote the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or $x \to -\infty$

optimization problems problems that are solved by finding the maximum or minimum value of a function

percentage error the relative error expressed as a percentage

propagated error the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

related rates are rates of change associated with two or more related quantities that are changing over time

relative error given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

rolle's theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ and if $f(a) = f(b),$ then there exists $c \in (a, b)$ such that $f'(c) = 0$

second derivative test suppose $f'(c) = 0$ and f'' is continuous over an interval containing $c;$ if $f''(c) > 0,$ then f has a local minimum at $c;$ if $f''(c) < 0,$ then f has a local maximum at $c;$ if $f''(c) = 0,$ then the test is inconclusive

tangent line approximation (linearization) since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

KEY EQUATIONS

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**
  $dy = f'(x)dx.$

KEY CONCEPTS

4.1 Related Rates

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

4.2 Linear Approximations and Differentials

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

$$dy = f'(x)dx$$

  is an approximation for the change in y. The actual change in y is

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the *propagated error*. The propagated error can be estimated by

$$dy \approx f'(x)dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

4.3 Maxima and Minima

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

4.4 The Mean Value Theorem

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = 0 = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is the Mean Value Theorem.

- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

4.5 Derivatives and the Shape of a Graph

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.
- If $f''(x) > 0$ over an interval I, then f is concave up over I.
- If $f''(x) < 0$ over an interval I, then f is concave down over I.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

4.6 Limits at Infinity and Asymptotes

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0$, $p(x)$ approaches ∞ or $-\infty$ at each end.
- For a rational function $f(x) = \dfrac{p(x)}{q(x)}$, the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \dfrac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

4.7 Applied Optimization Problems

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.

- Look for critical points to locate local extrema.

4.8 L'Hôpital's Rule

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function x^p, $p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function x^p, $p > 0$.

4.9 Newton's Method

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1, x_2, x_3, \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0, x_1, x_2, \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0, x_1, x_2, \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

4.10 Antiderivatives

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \; y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

CHAPTER 4 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

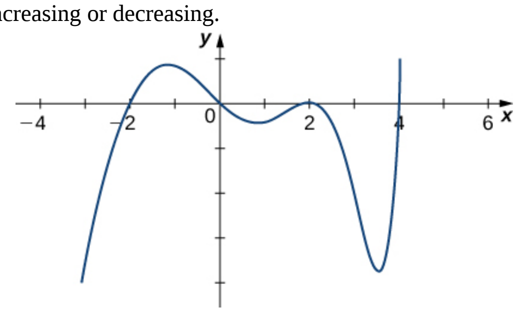
525. If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

526. If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

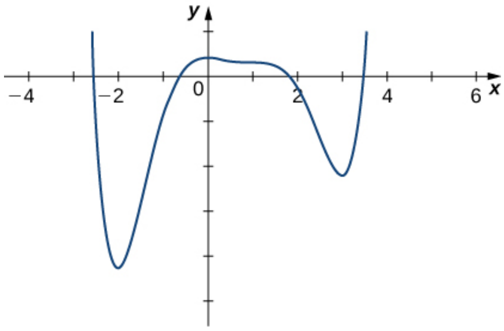
527. There is a function such that $f(x) < 0$, $f'(x) > 0$, and $f''(x) < 0$. (A graphical "proof" is acceptable for this answer.)

528. There is a function such that there is both an inflection point and a critical point for some value $x = a$.

529. Given the graph of f', determine where f is increasing or decreasing.



530. The graph of f is given below. Draw f'.



531. Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

532. Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

533. $f(x) = x + \sin^2(x)$ over $[0, \pi]$

534. $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

535. $x(t) = 3t^4 - 8t^3 - 18t^2$

536. $y = x + \sin(\pi x)$

537. $g(x) = x - \sqrt{x}$

538. $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

539. $\lim_{x \to \infty} \frac{3x\sqrt{x^2+1}}{\sqrt{x^4-1}}$

540. $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right)$

541. $\lim_{x \to 1} \frac{x-1}{\sin(\pi x)}$

542. $\lim_{x \to \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

543. $y = x^3 + 1,\ x_0 = 0.5$

544. $\frac{1}{x+1} = \frac{1}{2},\ x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

545. $g(x) = \sqrt{x} - \frac{1}{x^2}$

546. $f(x) = 2x + 6\cos x,\ F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

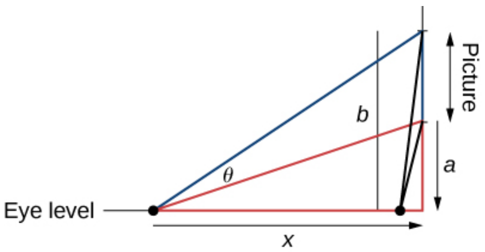
547. $y = \frac{1}{x(x+1)^2}$

548. $y = x - \sqrt{4 - x^2}$

549. A car is being compacted into a rectangular solid. The volume is decreasing at a rate of 2 m^3/sec. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

550. A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right)m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

551. The famous Regiomontanus' problem for angle maximization was proposed during the 15 th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



552. An airline sells tickets from Tokyo to Detroit for $\$1200$. There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?